3. Partitions of unity.

We define the *sum* of the non-negative real numbers r_i, $i \in I$, by the formula

$$\sum_{i\in I} r_i = \sup\left\{\sum_{j\in J} r_j : J \subset I \text{ and } J \text{ is finite}\right\}.$$

and we note that if $\sum_{i\in I} r_i < \infty$, then $r_i \neq 0$ for at most countably many $i \in I$.

A partition of unity of a topological space X is a collection $\mathcal{F} = \{f_i : i \in I\}$ of continuous functions $X \to [0,1]$ such that $\sum_{i\in I} f_i(x) = 1$ for every $x \in X$. The partition of unity $\mathcal{F}$ is *subordinated* to a cover $\mathcal{U}$ of X provided that the family $\{\mathrm{Supp}(f) : f \in \mathcal{F}\}$ is a refinement of $\mathcal{U}$.

1 Theorem *Every open cover of a pseudometrizable space has a subordinated partition of unity.*

Proof. Let $\mathcal{G}$ be an open cover of a pseudometric space (X, d). By the well-ordering theorem, we can write $\mathcal{G} = \{G_\alpha : \alpha < \lambda\}$, where λ is an ordinal number. We set $\sup\emptyset = 0$. For every $\alpha \leq \lambda$, the formula $g_\alpha(x) = \min\big(1, \sup_{\beta<\alpha} d(x, X \setminus G_\beta)\big)$ defines a function $g_\alpha : X \to [0,1]$; this fuction is continuous, because for all $x, y \in X$ we have that

$$|g_\alpha(x) - g_\alpha(y)| \leq \sup_{\beta<\alpha} |d(x, X \setminus G_\beta) - d(y, X \setminus G_\beta)| \leq d(x,y)\ .$$

For every $\alpha < \lambda$, we set $f_\alpha = g_{\alpha+1} - g_\alpha$, and we note that f_α is continuous $X \to [0,1]$. If a point $x \in X$ satisfies the inequality $f_\alpha(x) > 0$, then we have that $g_{\alpha+1}(x) > g_\alpha(x)$ and hence that $\sup_{\beta\leq\alpha} d\big(x, X \setminus G_\beta\big) > \sup_{\beta<\alpha} d\big(x, X \setminus G_\beta\big)$; as a consequence, we have that $d\big(x, X \setminus G_\alpha\big) > 0$. By the foregoing, we see that the support of f_α is contained in G_α.

We use transfinite induction to show that $\sum_{\beta<\alpha} f_\beta = g_\alpha$ for every $\alpha \leq \lambda$. The equation holds for $\alpha = 0$, because $\sum_{\beta<0} f_\beta \equiv 0 \equiv g_0$. Let $0 < \alpha \leq \lambda$ be such that the required equation holds for every $\gamma < \alpha$. Note that $g_\alpha = \sup_{\gamma<\alpha} g_{\gamma+1}$. For each $\gamma < \alpha$ we have that $f_\gamma = g_{\gamma+1} - g_\gamma$ and hence it follows from the inductive assumption that $g_{\gamma+1} = f_\gamma + g_\gamma = f_\gamma + \sum_{\beta<\gamma} f_\beta = \sum_{\beta\leq\gamma} f_\beta$. By the foregoing, we have that $g_\alpha = \sup_{\gamma<\alpha} \sum_{\beta\leq\gamma} f_\beta = \sum_{\beta<\alpha} f_\beta$. This completes the induction.

By the foregoing, we have that $\sum_{\alpha<\lambda} f_\alpha = g_\lambda$. Since $\mathcal{G}$ is an open cover, all values of g_λ are strictly positive. It follows that the collection $\mathcal{F} = \{f_\alpha/g_\lambda : \alpha < \lambda\}$ is a partition of unity. Moreover, $\mathcal{F}$ is subordinated to $\mathcal{G}$. □

The following result indicates the rôle of partitions of unity in the theory of pseudometrizable spaces.

2 Theorem *A space is pseudometrizable iff the space has the weak topology induced by some partition of unity.*

Proof. *Necessity.* Let (X, d) be a pseudometric space. By Theorem 1 there exists, for every $n \in \mathbb{N}$, a partition of unity $\{f_i : i \in I_n\}$ of X subordinated to the open cover $\{B_d(x, \frac{1}{n}) : x \in X\}$. We may assume that $I_n \cap I_k = \emptyset$ for $n \neq k$. We set $I = \bigcup_{n\in\mathbb{N}} I_n$, and we define functions g_i, $i \in I$, by setting $g_i = 2^{-n} f_i$ for $i \in I_n$. Then the collection $\mathcal{G} = \{g_i : i \in I\}$ is a partition of unity of X.

We denote by τ the weak topology induced on X by $\mathcal{G}$. Since $\mathcal{G}$ consists of τ_d-continuous functions, we have that $\tau \subset \tau_d$. To show that $\tau_d \subset \tau$, let $x \in X$ and $\epsilon > 0$. Choose $n \in \mathbb{N}$ so that $\frac{2}{n} < \epsilon$ and choose $i \in I_n$ so that $f_i(x) > 0$. Let $z \in X$ be such that $\mathrm{Supp}(f_i) \subset B_d(z, \frac{1}{n})$. Then we have that $x \in B_d(z, \frac{1}{n})$ and hence that $B_d(z, \frac{1}{n}) \subset B_d(x, \frac{2}{n})$. It follows from the foregoing that we habve $x \in \mathrm{Supp}(g_i) \subset B_d(x, \epsilon)$ for the the τ-open set $\mathrm{Supp}(g_i)$. We have shown that $\tau_d \subset \tau$.

Sufficiency. We assume that the topology τ of a space X is the weak topology induced by a partition of unity $\mathcal{G} = \{g_i : i \in I\}$. We define a pseudometric d of X by the formula

$$d(x, y) = \sup_{i\in I} |g_i(x) - g_i(y)| \,.$$

We show that $\tau_d = \tau$. To show that $\tau_d \subset \tau$, it suffices to show that, for all $x \in X$ and $r > 0$, we have that $B_d(x, r) \in \eta_x(\tau)$. Let $x \in X$ and $r > 0$. Since $\sum_{i\in I} g_i(x) = 1$, there exists a finite set $J \subset I$ such that $\sum_{i\in J} g_i(x) > 1 - \frac{r}{3}$. Set $n = |J|$. The functions g_i, $i \in J$, are τ-continuous, and hence there exists $V \in \eta_x(\tau)$ such that we have $|g_i(z) - g_i(x)| < \frac{r}{3n}$ for all $z \in V$ and $i \in J$. We show that $V \subset B_d(x, r)$. Let $v \in V$. Then we have that

$$\sum_{i\in J} g_i(v) > \sum_{i\in J} (g_i(x) - \tfrac{r}{3n}) > 1 - \tfrac{r}{3} - \tfrac{r}{3} = 1 - \tfrac{2r}{3}$$

and it follows, since $\sum_{i\in I} g_i(v) = 1$, that we have $g_i(v) < \frac{2r}{3}$ for every $i \in I \setminus J$. By the foregoing, we have that $|g_i(v) - g_i(x)| < \frac{2r}{3}$ for every $i \in I \setminus J$. It follows, since $|g_i(v) - g_i(x)| < \frac{r}{3}$ for every $i \in J$, that the inequality $d(v, x) = \sup_{i\in I} |g_i(v) - g_i(x)| \leq \frac{2r}{3}$ holds. As a consequence, we have that $V \subset B_d(x, r)$. Hence $B_d(x, r) \in \eta_x(\tau)$.

The family $\mathcal{C} = \{g_i^{-1}(O) : i \in I \text{ ja } O \Subset \mathbb{R}\}$ is a subbase of the topology τ. To prove the inclusion $\tau \subset \tau_d$, it suffices to show that $\mathcal{C} \subset \tau_d$. Let $i \in I$, $O \Subset \mathbb{R}$ and $x \in g_i^{-1}(O)$.

Since $g_i(x) \in O \Subset \mathbb{R}$, there exists $r > 0$ such that $(g(x) - r, g(x) + r) \subset O$. Now we have that $B_d(x, r) \subset g_i^{-1}(O)$, because if $d(y, x) < r$, then $|g_i(y) - g_i(x)| < r$ and hence $g_i(y) \in (g(x) - r, g(x) + r) \subset O$, which implies that $y \in g_i^{-1}(O)$. By the foregoing, we have that $g_i^{-1}(O) \in \tau_d$. □

We know (from Topology II) that a metric space (X, d) can be isometrically imbedded in the Banach space ℓ_X^∞ (this is the linear space of all bounded functions $X \to \mathbb{R}$, equipped with the supremum-norm). We can obtain an isometrism as follows. We fix a point x_0 of X, and for $x \in X$, we define $f_x : X \to \mathbb{R}$ be the formula $f_x(y) = d(y, x) - d(y, x_0)$. Then $x \mapsto f_x$ is the required isometrism $X \to \ell_X^\infty$.

As we mentioned in Example II.4.19, the Banach space ℓ_X^∞ is isometric with the linear space $C(\beta D)$ (equipped with the supremum-norm), where D is a discrete space with $|D| = |X|$. However, if we note that the functions f_x above are continuous on X, then we can embed X isometrically into a smaller Banach space: every f_x can be extended to continuous function $\bar{f}_x$ on βX, and the mapping $x \mapsto \bar{f}_x$ is an isometrism $X \to C(\beta X)$.

We can use Theorem 2 to show that every metrizable space X can be *topologically* embedded in a Banach space, which is in many ways much simpler than $C(\beta X)$. Before we define these Banach spaces, we extend the sum notation to cases where some of the summands may be negative. Let $r_i \in \mathbb{R}$, for $i \in I$, satisfy $\sum_{i \in I} |r_i| < \infty$. We set $I_+ = \{i \in I : r_i \geq 0\}$ and $I_- = \{i \in I : r_i < 0\}$, and we define

$$\sum_{i \in I} r_i = \sum_{i \in I_+} r_i - \sum_{i \in I_-} -r_i \ .$$

This generalized sum fulfills the usual rules for addition and gives a finite sum the same value as the ordinary sum.

Let A be a set. We define

$$\ell_A^2 = \Big\{ (x_\alpha)_{\alpha \in A} \in \mathbb{R}^A : \sum_{\alpha \in A} x_\alpha^2 < \infty \Big\}.$$

By the Schwarz Inequality, we have, for all $(x_\alpha), (y_\alpha) \in \ell_A^2$, that

$$\Big(\sum_{\alpha \in E} |x_\alpha y_\alpha| \Big)^2 \leq \sum_{\alpha \in E} x_\alpha^2 \cdot \sum_{\alpha \in E} y_\alpha^2$$

for every finite $E \subset A$. Hence we have that $\sum_{\alpha \in A} |x_\alpha y_\alpha| < \infty$. As a consequence, we can define a function $(\,.\,,\,.\,) : \ell_A^2 \times \ell_A^2 \to \mathbb{R}$ by the formula

$$\big((x_\alpha), (y_\alpha) \big) = \sum_{\alpha \in A} x_\alpha y_\alpha \ .$$

By the foregoing, we have for all $(x_\alpha), (y_\alpha) \in \ell^2_A$ that

$$\sum_{\alpha \in A} (x_\alpha + y_\alpha)^2 = \sum_{\alpha \in A} (x_\alpha^2 + 2x_\alpha y_\alpha + y_\alpha^2) = \sum_{\alpha \in A} x_\alpha^2 + 2\big((x_\alpha), (y_\alpha)\big) + \sum_{\alpha \in A} y_\alpha^2 < \infty .$$

Hence we can make ℓ^2_A into a linear space by equipping it with *pointwise operations*:

$$(x_\alpha) + (y_\alpha) = (x_\alpha + y_\alpha) \quad \text{and} \quad r(x_\alpha) = (rx_\alpha) .$$

It is easy to see that the function $(\cdot\,,\cdot)$ is an *inner product* of the linear space ℓ^2_A (i.e., it is symmetric, bilinear and satisfies the condition $(x, x) > 0$ for each $x \neq \bar{0}$). In this situation we can define a norm for the linear space ℓ^2_A by the formula

$$\|(x_\alpha)\| = \sqrt{((x_\alpha), (x_\alpha))} ,$$

in other words, by the formula $\|(x_\alpha)\|^2 = \sum_{\alpha \in A} x_\alpha^2$.

The foregoing gives us the following equation between the norm and the inner product:

$$\|(x_\alpha) + (y_\alpha)\|^2 = \|(x_\alpha)\|^2 + 2\big((x_\alpha), (y_\alpha)\big) + \|(y_\alpha)\|^2 .$$

The inner product space ℓ^2_A is a *generalized Hilbert space*; it is quite easy to show that the norm metric of ℓ^2_A is complete; hence ℓ^2_A, as a normed space, is a Banach space. The *unit sphere* of the space ℓ^2_A is the subset $S_{\ell^2_A} = \{\bar{x} \in \ell^2_A : \|\bar{x}\| = 1\}$.

3 Theorem *let X be a metrizable space. Then there exists a set A and an embedding $X \to S_{\ell^2_A}$.*

Proof. It follows from Theorem 2 that X has a partition of unity $\mathcal{F} = \{f_\alpha : \alpha \in A\}$ such that the weak topology induced by $\mathcal{F}$ coincides with the topology of X. We can define a mapping $\varphi : X \to S_{\ell^2_A}$ by the formula $\varphi(x)_\alpha = \sqrt{f_\alpha(x)}$. We show that φ is an embedding.

Since X is T_1 and $\mathcal{F}$ induces the topology of X, we see that φ is one-to-one. To show that φ is continuous, let $x_n \to x$ in X. We show that $\varphi(x_n) \to \varphi(x)$, in other words, that $\|\varphi(x_n) - \varphi(x)\| \to 0$. For all $a, b \geq 0$ we have that $\sqrt{ab} \leq \frac{1}{2}(a + b)$, and it follows that we have, for every $n \in \mathbb{N}$ that

$$\sum_{\alpha \in A} \sqrt{f_\alpha(x_n) f_\alpha(x)} \leq \frac{1}{2}\Big(\sum_{\alpha \in A} f_\alpha(x_n) + \sum_{\alpha \in A} f_\alpha(x)\Big) = 1 < \infty$$

and hence that

$$\|\varphi(x_n) - \varphi(x)\|^2 = \sum_{\alpha \in A} \left(\sqrt{f_\alpha(x_n)} - \sqrt{f_\alpha(x)} \right)^2 =$$

$$\sum_{\alpha \in A} f_\alpha(x_n) + \sum_{\alpha \in A} f_\alpha(x) - \sum_{\alpha \in A} 2\sqrt{f_\alpha(x_n) f_\alpha(x)} = 2 - 2 \sum_{\alpha \in A} \sqrt{f_\alpha(x_n) f_\alpha(x)}\,.$$

As a consequence, to show that $\varphi(x_n) \to \varphi(x)$, it suffices to show that $\sum_{\alpha \in A} \sqrt{f_\alpha(x_n) f_\alpha(x)} \to 1$ when $n \to \infty$. Let $\epsilon > 0$. Then there exists a finite set $\emptyset \neq B \subset A$ such that $\sum_{\alpha \in B} f_\alpha(x) > 1 - \frac{\epsilon}{2}$ and $f_\alpha(x) > 0$ for every $\alpha \in B$. We denote by m the number $|B|$ and by δ the minimum of the numbers $\frac{\epsilon}{2m}$ and $f_\alpha(x)$, $\alpha \in B$. As the functions f_α, $\alpha \in B$, are continuous, there exists $k \in \mathbb{N}$ such that $f_\alpha(x_n) > f_\alpha(x) - \delta^2$ for all $n \geq k$ and $\alpha \in B$. Now we have, for all $n \geq k$ and $\alpha \in B$, that

$$\sqrt{f_\alpha(x_n) f_\alpha(x)} > \sqrt{\big(f_\alpha(x) - \delta^2\big) f_\alpha(x)} \geq \sqrt{f_\alpha(x)^2 - \delta^2} \geq f_\alpha(x) - \delta \geq f_\alpha(x) - \tfrac{\epsilon}{2m}\,.$$

By the foregoing, we have, for every $n \geq k$, that

$$\begin{aligned}\sum_{\alpha \in A} \sqrt{f_\alpha(x_n) f_\alpha(x)} &\geq \sum_{\alpha \in B} \sqrt{f_\alpha(x_n) f_\alpha(x)} > \sum_{\alpha \in B} \big(f_\alpha(x) - \tfrac{\epsilon}{2m}\big) \\ &= \sum_{i \in B} f_\alpha(x) - \tfrac{\epsilon}{2} \geq 1 - \tfrac{\epsilon}{2} - \tfrac{\epsilon}{2} = 1 - \epsilon\,.\end{aligned}$$

Since we have, for every $\ell \in \mathbb{N}$, that $\sum_{\alpha \in A} \sqrt{f_\alpha(x_\ell) f_\alpha(x)} \leq 1$, we have shown that $\sum_{\alpha \in A} \sqrt{f_\alpha(x_n) f_\alpha(x)} \to 1$ and hence that $\varphi(x_n) \to \varphi(x)$.

Finally, let us note that also the mapping $\varphi^{-1} : \varphi(X) \to X$ is continuous: if x and x_n, $n \in \mathbb{N}$, are points of X such that $\varphi(x_n) \to \varphi(x)$, then we have for every $\alpha \in A$ that $\varphi(x_n)_i \to \varphi(x)_i$, i.e., that $f_\alpha(x_n) \to f_\alpha(x)$, and it follows, since X has the weak topology induced by $\{f_\alpha : \alpha \in A\}$, that we have $x_n \to x$. □

The proofs of Theorems 2 and 3 show that, if a T_1-space X has the weak topology induced by a partition of unity $\{f_\alpha : \alpha \in A\}$, then the mapping $x \mapsto \langle f_\alpha(x) \rangle_{\alpha \in A}$ gives an embedding of X in ℓ^∞_A and the mapping $x \mapsto \langle \sqrt{f_\alpha(x)} \rangle_{\alpha \in A}$ gives an embedding of X in ℓ^2_A. We mention (without proof) that the first mapping also gives an embedding of X into (the unit sphere of) the Banach space ℓ^1_A, where ℓ^1_A is the linear space of all "summable" functions $A \to \mathbb{R}$, equipped with norm $||h|| = \sum_{\alpha \in A} |h(\alpha)|$.

We have seen above that pseudometrizability of a space can be characterized in terms of the existence of a certain partition of unity. Next we note that also full normality of a space can be characterized in terms of the existence of partitions of unity.

4 Theorem *A space is fully normal iff every open cover of the space has a subordinated partition of unity.*

Proof. *Necessity.* Let X be fully normal space, and let $\mathcal{U}$ be an open cover of X. By Theorem 1.8, there exists a continuous pseudometric d of X and a τ_d-open cover $\mathcal{V}$ which refines $\mathcal{U}$. By Theorem 1, the pseudometric space (X, d) has a partition of unity F which is subordinated to $\mathcal{V}$. Since d is continuous on X, the collection F is also a partition of unity of the space X. Moreover, since $\mathcal{V}$ refines $\mathcal{U}$, the partition of unity F is subordinate to $\mathcal{U}$.

Sufficiency. Assume that every open cover of X has a subordinated partition of unity. To show that X is fully normal, let $\mathcal{U}$ be an open cover of X. Let F be a partition of unity of X which is subordinate to $\mathcal{U}$. By Theorem 2, there exists a pseudometric d of X such that the weak topology induced on X by F coincides with τ_d. Since every $f \in F$ is continuous on X, the weak topology induced by F is coarser than the topology of X. As a consequence, d is a continuous pseudometric of X. Moreover, the τ_d-open cover $\{\mathrm{Supp}(f) : f \in F\}$ of X refines $\mathcal{U}$.

We have shown that every open cover of X has a τ_d-open refinement for some continuous pseudometric of X. By Theorem 1.8, the space X is fully normal. □

Before we turn to consider the next topic of "continuous selections", we observe that partitions of unity have certain properties of locally finitely supported families which can be used to construct partitions of unity which actually are locally finitely supported.

5 Lemma *Let $\{f_\alpha : \alpha \in A\}$ be a partition of unity of X. For all $x \in X$ and $\epsilon > 0$, there exists $V \in \eta_x$ and a finite $E \subset A$ such that $\sum_{\alpha \in A \smallsetminus E} f_\alpha(z) < \epsilon$ for every $z \in V$.*

Proof. Exercise. □

6 Lemma *Let $\{f_\alpha : \alpha \in A\}$ be a partition of unity of X and let $B \subset A$. Then the functions $\sum_{\alpha \in B} f_\alpha$ and $\sup_{\alpha \in B} f_\alpha$ are continuous.*

Proof. Exercise. □

7 Proposition *Let $\{f_\alpha : \alpha \in A\}$ be a partition of unity of X. Then there exists a partition of unity $\{g_\alpha : \alpha \in A\}$ of X such that we have $\mathrm{Supp}(g_\alpha) \subset \mathrm{Supp}(f_\alpha)$ for every $\alpha \in A$ and the family $\{\mathrm{Supp}(g_\alpha) : \alpha \in A\}$ is locally finite (as an indexed family).*

Proof. By Lemma 6, the function $h = \sup_{\alpha \in A} f_\alpha$ is continuous. It follows that, for every $\alpha \in A$, the set $U_\alpha = \{x \in X : f_\alpha(x) > \frac{1}{2}h(x)\}$ is open. Note that the family

$\mathcal{U} = \{U_\alpha : \alpha \in A\}$ covers X. We show that $\mathcal{U}$ is locally finite. Let $x \in X$. Then there exists $\alpha_x \in A$ such that $f_{\alpha_x}(x) > \frac{3}{4}h(x)$. Denote by V the nbhd $\{z \in X : f_{\alpha_x}(z) > \frac{3}{4}h(x)\}$. By Lemma 5, there exists $W \in \eta_x$ and a finite $B \subset A$ such that we have $f_\alpha(z) < \frac{1}{4}h(x)$ for all $z \in W$ and $\alpha \in A \setminus B$. We show that $\{\alpha \in A : U_\alpha \cap V \cap W \neq \emptyset\} \subset B$. Assume on the contrary that there exist $\alpha \in A \setminus B$ and $z \in U_\alpha \cap V \cap W$. Then we have that $f_\alpha(z) > \frac{1}{2}h(z)$, $f_{\alpha_x}(z) > \frac{3}{4}h(x)$ and $f_\alpha(z) < \frac{1}{4}h(x)$. As a consequence, we have that $\frac{1}{4}h(x) > f_\alpha(z) > \frac{1}{2}h(z)$ and $h(z) \geq f_{\alpha_x}(z) > \frac{3}{4}h(x)$, but this is a a contradiction. It follows from the foregoing that $\mathcal{U}$ is locally finite (as an indexed family).

For every $\alpha \in A$, the function $k_\alpha = 0 \vee (f_\alpha - \frac{1}{2}h)$ is continuous and $\mathrm{Supp}(k_\alpha) = U_\alpha$. Hence the collection $\{k_\alpha : \alpha \in A\}$ is locally finitely supported, and it follows that the function $k = \sum_{\alpha \in A} k_\alpha$ is continuous. If we set $g_\alpha = k_\alpha / k$ for each $\alpha \in A$, then we get a partition of unity $\{g_\alpha : \alpha \in A\}$ with the required properties. □

Next we shall use partitions of unity to prove a fundamental result in the "theory of selections". We need some definitions before can state the result.

Let φ be a mapping from a space X into the family of all non-empty subsets of a space Y. We say that φ is a *carrier*. A *selection* for the carrier φ is a mapping $f : X \to Y$ such that we have $f(x) \in \varphi(x)$ for every $x \in X$; if f is continuous, then we say that f is a *continuous selection* for φ.

8 Example Extensions of mappings can be considered as selections. Let $A \subset X$ and $g : A \to Y$. If we define $\varphi : X \to \mathcal{P}(Y)$ by setting

$$\varphi(x) = \begin{cases} g\{x\} & \text{, for } x \in A \\ Y & \text{, for } x \in X \setminus A, \end{cases}$$

then a (continuous) selection for φ is a (continuous) extension of g. □

Let $\varphi : X \to \mathcal{P}(Y)$ be a carrier. We say that φ is *lower semi-continuous (lsc)* provided that, for every $G \Subset Y$, we have that $\{x \in X : \varphi(x) \cap G \neq \emptyset\} \Subset X$.

9 Examples (a) To every mapping $f : X \to Y$ we can associate the carrier $x \mapsto f\{x\}$. We denote also this carrier by f. The carrier f is lsc iff we have $f^{-1}(G) \Subset X$ for each $G \Subset Y$, in other words, iff the mapping f is continuous.
(b) Let f be an open and continuous mapping $X \to Y$. Then the formula $\varphi(x) = f^{-1}f\{x\}$ defines an lsc carrier $\varphi : X \to \mathcal{P}(X)$. To see this, let $G \Subset X$. Since f is open and continuous, we have that $f^{-1}(f(G)) \Subset X$. Moreover, we have that $f^{-1}(f(G)) = \{x \in X : \varphi(x) \cap G \neq \emptyset\}$. □

For the proof of the basic selection theorem, we need two auxiliary results. The first of these is a technical result, but the second one is already a "near selection theorem".

10 Lemma *Let φ and ψ be lsc carriers $X \to \mathcal{P}(Y)$, let d be a continuous pseudometric of Y and let $r > 0$ be such that we have $d(\varphi(x), \psi(x)) < r$ for every $x \in X$. Then the formula $\theta(x) = B_d(\varphi(x), r) \cap \psi(x)$ defines an lsc carrier $\theta : X \to \mathcal{P}(Y)$.*

Proof. To show that θ is lsc, let $G \Subset Y$ and let x be a point of the set $\{z \in X : \theta(z) \cap G \neq \emptyset\}$. Let $u \in \theta(x) \cap G$. Then $u \in B_d(\varphi(x), r)$ and hence there exists $\delta > 0$ such that $\varphi(x) \cap B_d(u, r - \delta) \neq \emptyset$. Since φ is lsc, the set $V = \{z \in X : \varphi(z) \cap B_d(u, r - \delta) \neq \emptyset\}$ is a nbhd of x. We also have that $u \in \psi(x)$, and it follows, since ψ is lsc, that the set $W = \{z \in X : \psi(z) \cap G \cap B_d(u, \delta)\}$ is a nbhd of x. We show that $V \cap W \subset \{z \in X : \theta(z) \cap G \neq \emptyset\}$. Let $z \in V \cap W$. Since $z \in V$, there exists $v \in \psi(z) \cap G$ such that $d(v, u) < \delta$. Since $z \in W$, there exists $w \in \varphi(z)$ such that $d(w, u) < r - \delta$. Now we have that $d(v, \varphi(z)) \leq d(v, w) \leq d(v, u) + d(u, w) < \delta + r - \delta = r$ and hence we have that $v \in \theta(z)$, and further, that $\theta(z) \cap G \neq \emptyset$. We have shown that $V \cap W \subset \{z \in X : \theta(z) \cap G \neq \emptyset\}$. By the foregoing, we have that $\{z \in X : \theta(z) \cap G \neq \emptyset\} \Subset X$. □

11 Lemma *Let X be a fully normal space, Y a normed linear space and $\varphi : X \to \mathcal{P}(Y)$ an lsc carrier such that every $\varphi(x)$ is convex. For every $\epsilon > 0$, there exists a continuous mapping $f : X \to Y$ such that we have, for every $x \in X$, that $d(f(x), \varphi(x)) < \epsilon$, where d denotes the norm distance in Y.*

Proof. For every $y \in Y$, let $U_y = \{x \in X : \varphi(x) \cap B_d(y, \epsilon) \neq \emptyset\}$, and note that we have $U_y \Subset X$ because φ is lsc. By Theorem 4 and Proposition 7, the open cover $\mathcal{U} = \{U_y : y \in Y\}$ of X has a subordinated locally finitely supported partition of unity $\{g_\alpha : \alpha \in A\}$. For every $\alpha \in A$, let $y_\alpha \in Y$ be such that $\mathrm{Supp}(g_\alpha) \subset U_{y_\alpha}$. Define $f : X \to Y$ by setting

$$f(x) = \sum_{\alpha \in A} g_\alpha(x) y_\alpha \,.$$

Since $\{g_\alpha : \alpha \in A\}$ is locally finitely supported, the function f is continuous. To show that f has the required property, let $x \in X$. Denote by B the finite set $\{\alpha \in A : g_\alpha(x) \neq 0\}$. For every $\alpha \in B$, we have that $x \in \mathrm{Supp}(g_\alpha) \subset U_{y_\alpha}$ and hence that $d(y_\alpha, \varphi(x)) < \epsilon$. As a consequence, the point $f(x) = \sum_{\alpha \in B} g_\alpha(x) y_\alpha$ of Y is a convex combination of points from the set $B_d(\varphi(x), \epsilon)$. Since the set $\varphi(x)$ is convex, also the set $B_d(\varphi(x), \epsilon)$ is convex. By the foregoing, we have that $f(x) \in B_d(\varphi(x), \epsilon)$. □