QCD thermodynamics at 4-loop order: quark number susceptibilities

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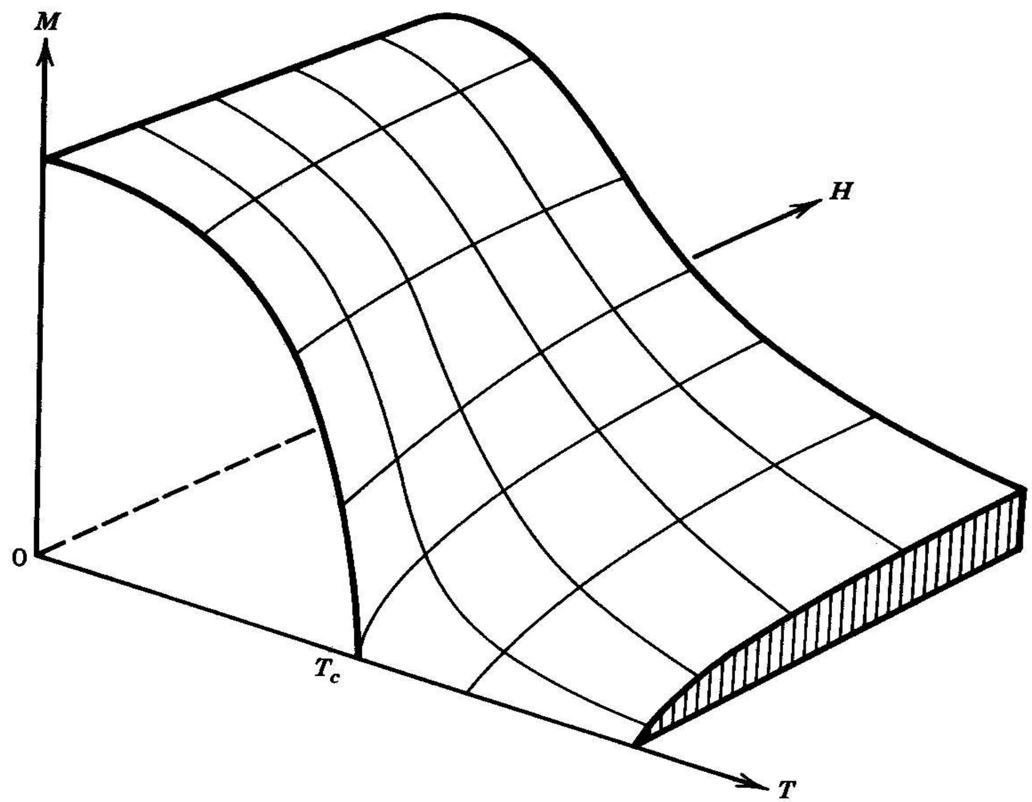
Outline of talk

- Introduction and motivation
  - What does 'susceptibility' mean?
  - Why compute quark number susceptibilities in pQCD?
- Evaluation of the QCD grand potential with non-zero μ
  - Basics of finite T field theory
  - Diagrammatic expansion of $\ln Z$ and IR problems
  - Effective three-dimensional theories
  - Effects of finite μ on the computations
- Quark number susceptibilities
  - Results from the perturbative evaluation of $\ln Z$
  - Lattice approach
- The grand potential at large μ/T
  - T$=$0: the Freedman-McLerran result
  - Problems with expanding in small T/μ

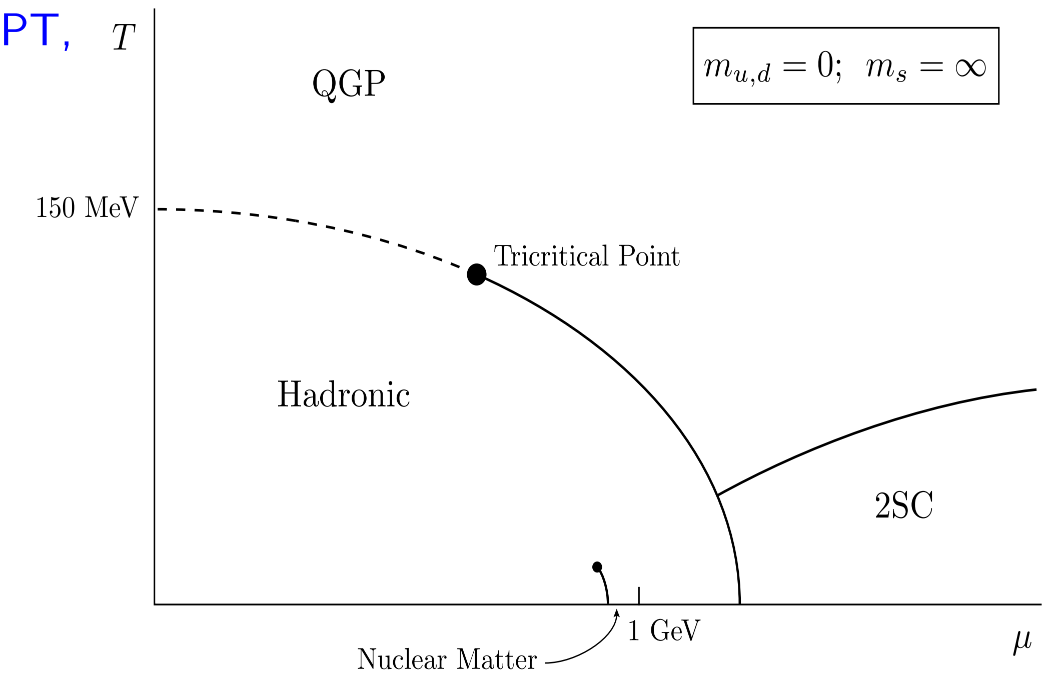
What is susceptibility?

- In general 'susceptibility' $\approx$ 'response function'
  - E.g. electric susceptibility $\chi \sim \frac{\boldsymbol{P}}{\boldsymbol{E}}$
- System perturbed linearly by external field $\phi \Rightarrow \hat{H}_0 \rightarrow \hat{H}_0 - \hat{O}\phi \Rightarrow \chi \equiv \frac{\partial\langle\hat{O}\rangle}{\partial\phi} \sim \frac{\partial^2 \ln Z}{\partial\phi^2}$
- Magnetic susceptibility: magnetization M induced by external magnetic field H
  - $\mathrm{d}G = -S\mathrm{d}T - M\mathrm{d}H$
  - $M = -\frac{\partial G}{\partial H}$
  - $\chi = \frac{1}{V}\frac{\partial M}{\partial H}$
  - Even if $M = 0$, $\chi \neq 0$
  - $\chi|_{H=0}$ describes the behaviour of G at small H



QCD and finite baryon density

- In QCD no mixing of flavours $\Rightarrow$ net density of flavour $n_f \equiv \frac{1}{V}\left(N_f - N_{\bar{f}}\right)$ conserved $\forall f$
  - Introduce corresponding chemical potentials μ_f
- $\langle n \rangle = T \frac{\partial \ln Z}{\partial \mu} \sim \mu \Rightarrow$ 'finite μ' $\approx$ 'finite n'
- Quark number susceptibilities $\chi_{fg} \equiv \frac{\partial \langle n_f \rangle}{\partial \mu_g} = \frac{T}{V} \frac{\partial^2 \ln Z}{\partial \mu_f \partial \mu_g} \sim \langle n_f n_g \rangle - \langle n_f \rangle \langle n_g \rangle$
  - Computable on lattice at $\mu = 0$ while directly relating to finite density!
  - Susceptibilities of conserved charges probe deconfinement in heavy ion collisions
  - So far known only to $\mathcal{O}(g^4 \ln g)$ in PT, where in direct contradiction with lattice data
  - Here theory ahead of experiment



Basics of thermodynamics

- Statistical mechanics: partition function gives all thermodynamic quantities
  - $Z \equiv \mathrm{Tr}\,\rho \equiv \mathrm{Tr}\exp[-\beta\,(H-\mu N)]$
  - $P = T\frac{\partial \ln Z}{\partial V}$
  - $S = \frac{\partial(T \ln Z)}{\partial T}$
  - $N = T\frac{\partial \ln Z}{\partial \mu}$
  - $U = -PV + TS + \mu N$
- $Z = \int \mathrm{d}\phi \,\langle\,\phi\,|\,\exp[-\beta\,(H-\mu N)]\,|\,\phi\,\rangle \Rightarrow$ natural path integral representation for Z:
  - $Z = \int \mathcal{D}\phi \exp\Big\{-\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d x\ (\mathcal{L}-\mu N)\Big\}$, where ϕ periodic/antiperiodic in τ
- In PT this leads to evaluation of vacuum diagrams with finite T Feynman rules: replace
  - $\int\frac{\mathrm{d}^d p}{(2\pi)^d} \to T\sum_{p_0}\int\frac{\mathrm{d}^{d-1}p}{(2\pi)^{d-1}}$
    - $p_0 = 2n\pi T$ for bosons
    - $p_0 = (2n+1)\pi T - i\mu$ for fermions
  - $\delta^{(4)}(p) \to \beta(2\pi)^{-1}\delta^{(3)}(\boldsymbol{p})\delta_{p_0,0}$

QCD pressure

- For Euclidean QCD with massless quarks $\mathcal{L} - \mu N = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \bar{\psi} \not{D} \psi - \psi^\dagger \boldsymbol{\mu} \psi$, where
  - $\psi = (\psi_1, \psi_2, \ldots, \psi_{n_f})$,
  - $\boldsymbol{\mu} = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_{n_f})$
- How to evaluate $\ln Z_{\mathrm{QCD}} \sim p_{\mathrm{QCD}}$?
  - Lattice simulations
    - Only fundamentally non-perturbative analytic tool available
    - Works only for $\mu = 0$ and $T \lesssim$ a few T_c
  - Perturbation theory: done up to
    - $\mathcal{O}(g^6 \ln g)$ at $\mu = 0$ [1]
    - $\mathcal{O}(g^6 \ln g)$ at $0 \leq \mu \lesssim 4T$ [2]
    - $\mathcal{O}(g^4)$ at $T = 0$ [3]
- Results for finite μ give susceptibilities
  - Interesting limit $\mu \to 0$, where χ_{fg} measurable on lattice
  - At $\mu = 0$ symmetry between massless flavours $\Rightarrow \chi_{fg} = \chi \delta_{fg} + \widetilde{\chi} (1 - \delta_{fg})$

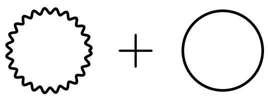
[1] Kajantie et al. hep-ph/0211321

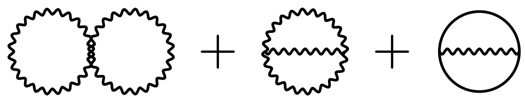
[2] AV hep-ph/0212283 + in preparation

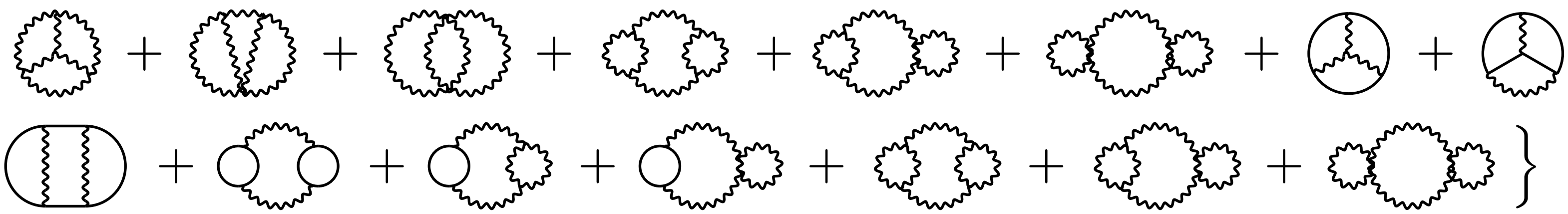
[3] Freedman, McLerran, Phys. Rev. D 16 (1977) 1130

Diagrammatic expansion of $\ln Z_{\text{QCD}}$

- Straightforward expansion of $\ln Z$ in terms of g gives at 3-loop order (without ghosts):

$p_1 =$  $= \frac{8\pi^2 T^4}{45}\left(1+\frac{21 n_f}{32}\right)+\frac{T^2}{2}\sum_f \mu_f^2+\frac{1}{4\pi^2}\sum_f \mu_f^4$

$p_2 = g^2\{$  $\}$

$p_3 = g^4\{$  $\}$

- Diagrams have been computed for $\mu = 0$ [4] and later for arbitrary μ [5]
- However, this is not enough: A_0 acquires mass m_{el} at 1-loop order $\Rightarrow$
  - screening to regulate IR divergencies
    - A_μ-propagator $\frac{g_{\mu\nu}}{P^2} \longrightarrow \frac{g_{\mu\nu}-\delta_{\mu 0}\delta_{\nu 0}}{P^2}+\frac{\delta_{\mu 0}\delta_{\nu 0}}{P^2+m^2}$
    - in coordinate space $\frac{1}{4\pi r} \longrightarrow \frac{e^{-mr}}{4\pi r}$
  - $\mathcal{O}(g^3)$ and $\mathcal{O}(g^4 \ln g)$ contributions to p_{QCD} through resummation

[4] Arnold, Zhai, Phys. Rev. D **50** (1994) 7603 [hep-ph/9408276]; Phys. Rev. D **51** (1995) 1906 [hep-ph/9410360].
[5] AV, in preparation

- Continuing to even higher orders in the expansion get more IR-problems: non-perturbative contributions from scale g^2T enter at $\mathcal{O}(g^6)$ (Linde)
  - Infinite number of diagrams should be computed at this order
  - Need some way to isolate the non-perturbativity
- Another approach to resummation: dimensional reduction and effective theories[6]
  - Fermions and non-zero Matsubara modes of bosons decouple at high $T \Rightarrow$
  - Integrate out these modes to get effective 3d theories for the soft scales $\Rightarrow$
  - Get separated from the perturbative expansion all
    - Contributions of scales gT and g^2T
    - Terms non-analytic in α_s
- Scale gT contributes (perturbatively) through 3d Yang-Mills + adjoint Higgs theory
  - $\mathcal{L}_{\mathsf{E}} \;=\; \frac{1}{2}\mathrm{Tr}\, F_{ij}^2 + \mathrm{Tr}\,[D_i, A_0]^2 + m_{\mathsf{el}}^2 \mathrm{Tr}\, A_0^2 + \frac{ig^3}{3\pi^2}\sum_f \mu_f\, \mathrm{Tr}\, A_0^3 + \lambda_{\mathsf{E}}^{(1)}(\mathrm{Tr}\, A_0^2)^2 + \lambda_{\mathsf{E}}^{(2)}\mathrm{Tr}\, A_0^4 + \delta\mathcal{L}_{\mathsf{E}}$
- Non-perturbativity isolated to 3d pure YM theory corresponding to scale g^2T
  - $\mathcal{L}_{\mathsf{M}} \;=\; \frac{1}{2}\mathrm{Tr}\, F_{ij}^2 + \delta\mathcal{L}_{\mathsf{M}}$
  - Contribution to p independent of μ at $\mathcal{O}(g^6) \Rightarrow$ no non-perturbative contributions to χ at this order

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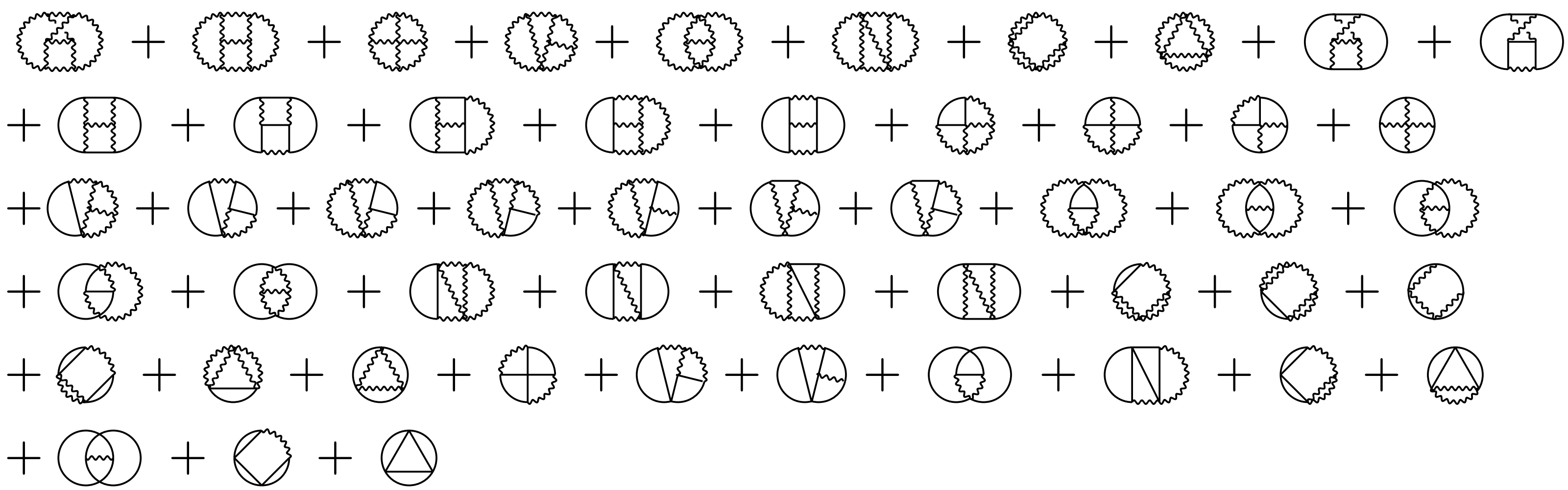
[6] Ginsparg, Nucl. Phys. B 170 (1980) 388; Appelquist, Pisarski, Phys. Rev. D 23 (1981) 2305

- To get p_{QCD} up to $\mathcal{O}(g^6 \ln g)$ need to compute in the effective theories
  - All 3-loop diagrams
  - Logarithmically divergent parts of all 4-loop diags (done by Y. Schröder)
    - ∗ Skeletons alone read



- Even finite parts of these currently being calculated to very high precision[7]
  - Contract the tensorial indices using Form[8] (a single diagram may contain $\sim 10^7$ terms)
  - Use tricks (changes of integration momenta, partial integration, ...) to reduce the number of independent integrals to a set of $\mathcal{O}(10)$ master integrals
  - Compute the master integrals analytically or numerically

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[7] Schröder (+ AV), in progress

[8] Vermaseren, math-ph/0010025; http://www.nikhef.nl/∼form

Effects of small but finite μ

- Need to generalize the computation of the full theory vacuum diagrams to non-zero μ
  - Same techniques work as at $\mu = 0$: Fourier-transform propagators into x-space using

$$\int \frac{\mathrm{d}^3\mathrm{q}}{(2\pi)^3} \frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{\mathbf{q}^2 + (q_0 - i\mu)^2} = \frac{\exp[-\left(|q_0| - i\mu\, \mathrm{sign}\, q_0\right) r]}{4\pi T}$$

  - Calculations tedious and lengthy - yet mostly straightforward

- 3d effective theory describing scale gT modified
  - Finite μ -corrections to the parameters of the theory[9]
  - New term $\frac{ig^3}{3\pi^2} \sum_f \mu_f \,\mathrm{Tr}\, A_0^3$ appears in $\mathcal{L} \Rightarrow$ new diagrams to evaluate

- Result expressible in terms of functions
  - $\aleph(n, z) \equiv \zeta'(-n, z) + (-1)^{n+1} \zeta'(-n, z^*),\ n = 0, 1, 2, 3$
  - $\aleph(z) \equiv \Psi(z) + \Psi(z^*)$

  with $z = 1/2 - i\mu/(2\pi T)$

- μ/T-expansion of the result gives $p \sim T^4 + \mu^2 T^2 + \ldots$
  - χ_{fg} at $\mu = 0$ directly available by definition

[9] Hart, Laine, Philipsen, Nucl. Phys. B **586** (2000) 443 [hep-ph/0004060].

Quark number susceptibilities

- Up to $\mathcal{O}(g^6)$ the diagonal susceptibility χ reads in PT (here $n_f = \#$ of flavours)

$$
\begin{aligned}
\frac{\chi}{\chi_0}|_{\mu=0} = {} & 1 - 2\,\frac{g^2(\rho)}{4\pi^2} + 8\sqrt{1+\frac{n_f}{6}}\left(\frac{g^2(\rho)}{4\pi^2}\right)^{3/2} + 12\left(\frac{g^2(\rho)}{4\pi^2}\right)^2 \ln\frac{g^2(\rho)}{4\pi^2} \\
- & \frac{1}{36}\left\{12\left(33-2n_f\right)\ln\frac{e^\gamma\rho}{4\pi T} - 432\ln\left(1+\frac{n_f}{6}\right) + 133 + 26\,n_f + 16\left(17+2\,n_f\right)\ln 2 - 432\gamma - 432\frac{\zeta'(-1)}{\zeta(-1)}\right\}\left(\frac{g^2(\rho)}{4\pi^2}\right)^2 \\
+ & \frac{1}{12\sqrt{1+n_f/6}}\left\{4\left(6+n_f\right)\left(33-2\,n_f\right)\ln\frac{e^\gamma\rho}{4\pi T} - 669 + 30\,n_f + 4\,n_f^2 - 36\pi^2 + 4\left(99-24\,n_f-4\,n_f^2\right)\ln 2\right. \\
+ & \left.\frac{28}{9}\left(6+n_f\right)^2\zeta(3)\right\}\left(\frac{g^2}{4\pi^2}\right)^{5/2} \\
+ & 2\left\{2\left(33-2n_f\right)\ln\frac{e^\gamma\rho}{4\pi T} + \frac{59}{9} + n_f - 8\,n_f\ln 2 + \frac{7}{3}\left(6+n_f\right)\zeta(3)\right\}\left(\frac{g^2}{4\pi^2}\right)^3\ln\frac{g^2}{4\pi^2} \\
- & \frac{33-2\,n_f}{108}\left\{6\left(33-2n_f\right)\ln\frac{e^\gamma\rho}{4\pi T} - 432\ln\left(1+\frac{n_f}{6}\right) + 88 + 26\,n_f + 16\left(17+2\,n_f\right)\ln 2 - 432\gamma\right. \\
- & \left.432\frac{\zeta'(-1)}{\zeta(-1)} - \frac{2889}{33-2\,n_f}\right\}\left(\frac{g^2}{4\pi^2}\right)^3\ln\frac{e^\gamma\rho}{4\pi T} + C(n_f)\left(\frac{g^2}{4\pi^2}\right)^3 + \mathcal{O}(g^7)
\end{aligned}
$$

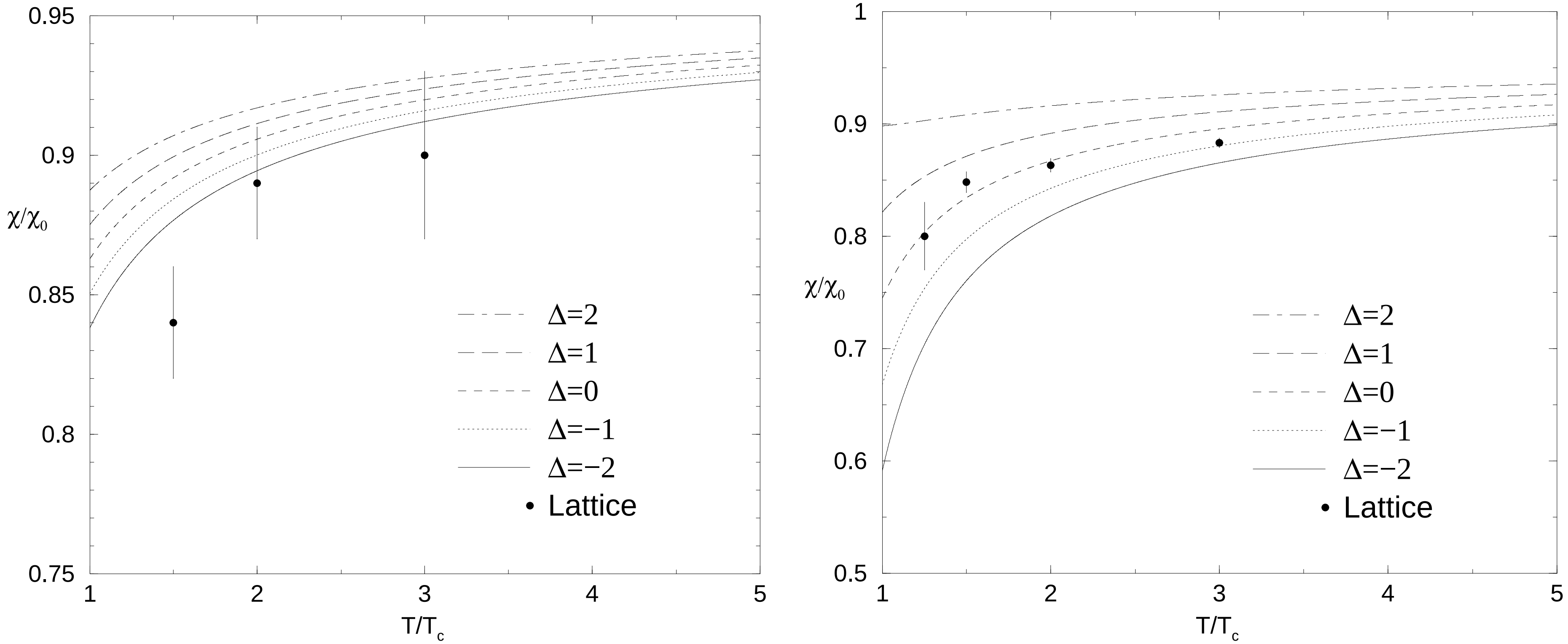
- $C(n_f)$ remains undetermined - yet fully perturbative
  - Would require full 4-loop calculation
  - Can be treated as a variational parameter here

- Recently susceptibilities under extensive study also on the lattice[10] [11] [12]
  - Motivation: probing QGP in heavy ion collisions and testing different resummation schemes in PT
  - Only applicable for $T/T_c \lesssim 3$ and $\mu = 0$
  - Simulations performed for $N_f = 2,\ 4$ and quenched QCD
  - Results show significant departure from ideal gas result - also inconsistency with HTL approach observed
  - Unfortunately results seem to vary with time
- PT result for χ well compatible with present lattice data - however dependence on $C(n_f)$ too strong for accurate predictions
  - Perturbativity of the susceptibilities anyhow plausible
- Convergence properties of perturbation expansion apparently good already at $T_c < T \lesssim 3T_c$
  - Effective theory however well-defined only for higher T

[10] Gavai, Gupta, hep-lat/0211015

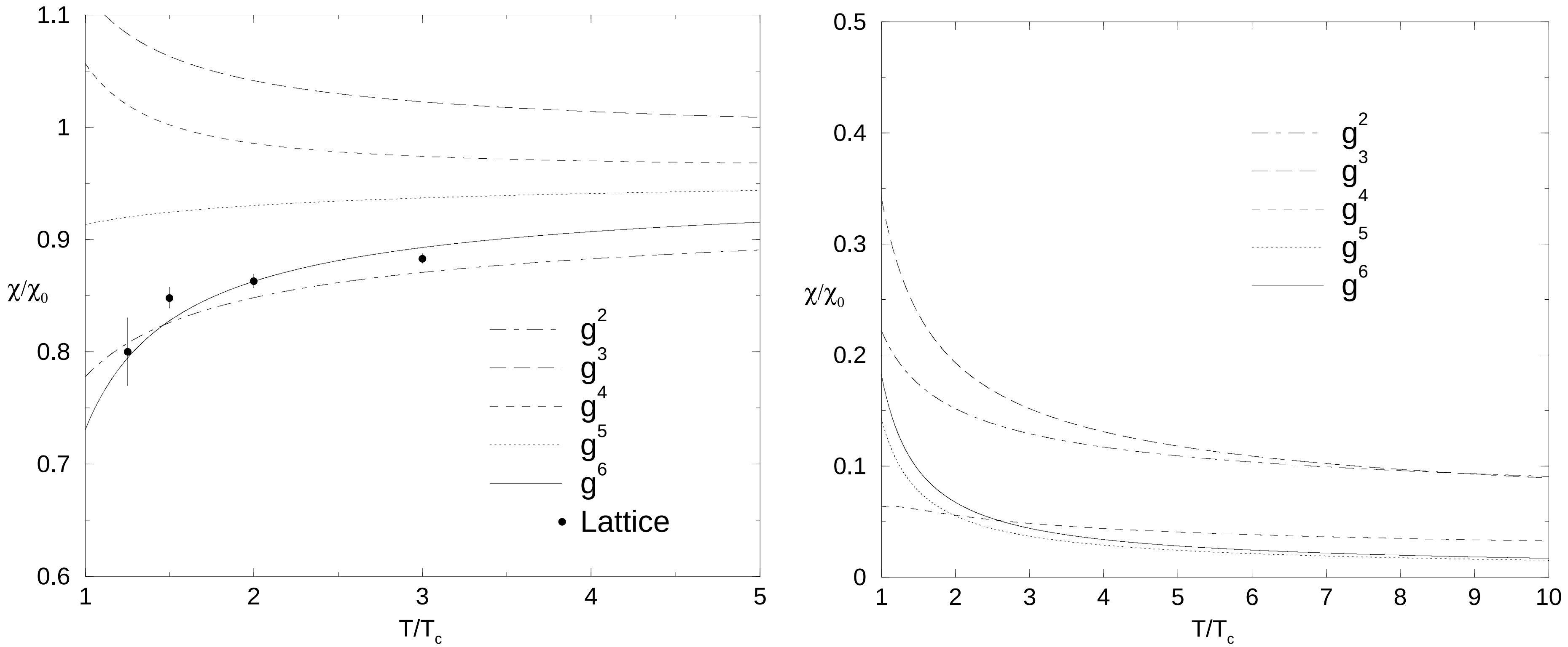
[11] Gavai, Gupta, Majumdar, Phys. Rev. D **65** (2002) 054506 [hep-lat/0110032]

[12] Bernard *et al.* [MILC Collaboration], [hep-lat/0209079]



Left: PT expression of χ/χ_0 at $n_f = 0$ plotted against the results of [8] for different values of $C(0)$. In [8] the results were obtained using staggered quarks in quenched QCD (no quark loops) with continuum extrapolation.

Right: Perturbative result at $n_f = 2$ together with lattice data from [9]. No continuum extrapolation in [9].



Left: χ/χ_0 at $n_f = 2$ plotted to different perturbative orders for $C(2) = -45$. Lattice data is again from [9].
Right: The absolute values of the individual terms of the series.

What next? Grand potential at small T, large μ

- Effective theories for high T work only for[13] $\mu/T \lesssim 4$
- No effective lower-dimensional theory exists for low T
  - Fermions give dominant contribution to diagrams
- At $T = 0$ grand potential known (numerically) up to $\mathcal{O}(g^4)$[14]
  - Analytic calculation underway[15]
  - Need diagrams of full theory up to 3-loop order + perform resummation
  - Problems with IR behaviour in the limit $T \to 0$
- Small T/μ corrections to F-McL result problematic
  - How to deal with such expressions as $g^4 \exp[-g\mu/T]$?

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13 Hart, Laine, Philipsen, Nucl. Phys. B **586** (2000) 443 [hep-ph/0004060].

14 Freedman, McLerran, Phys. Rev. D **16** (1977) 1169

15 AV, in progress

Summary

- Grand potential of QCD has been computed in the QGP phase to $\mathcal{O}(g^6 \ln g)$ with finite μ
  - Restrictions: $T \gg T_c$ and $\mu/T \lesssim 4$
- Diagonal quark number susceptibility computed to same order and compared with lattice data
  - Good agreement - however result still contains (perturbative) uncertainties
- Next order computation both for grand potential and susceptibilities extremely laborious
  - Still should be done at some point
- $T = 0$ result obtainable to $\mathcal{O}(g^4)$ - however small T/μ corrections more problematic