The pressure of QCD at finite temperatures and chemical potentials

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Outline of talk

- Introduction and motivation
  - The QCD phase diagram
  - Why drive perturbation theory to four-loop order?

- Perturbation theory at high T
  - Dimensional reduction
  - Quark number susceptibilities
  - Lattice tests

- Perturbation theory at high μ
  - Summation of ring diagrams
  - $T = 0$: the Freedman-McLerran result
  - Small but non-zero T?

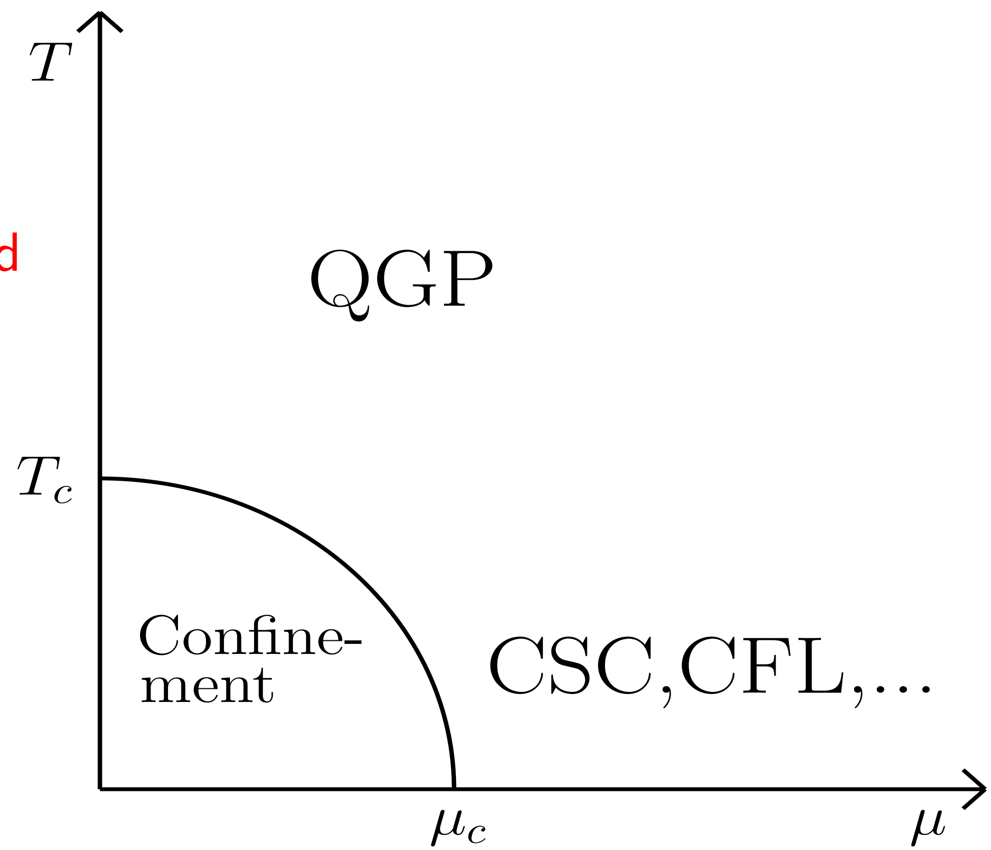
- Conclusions and future directions

QCD and finite baryon density

- Study QCD with n_f flavors of massless quarks keeping N_c arbitrary
  - $\mathcal{L} - \mu N = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \bar{\psi} \not{D} \psi - \psi^\dagger \boldsymbol{\mu} \psi$
    - $\psi = (\psi_1, \psi_2, \ldots, \psi_{n_f})$,
    - $\boldsymbol{\mu} = \mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_{n_f})$

- Here μ_f independent $\forall f$, since the net density of each flavor conserved

- In free theory $\langle N_f \rangle = T \frac{\partial \ln Z}{\partial \mu_f}$ $\sim \mu_f \Rightarrow$ 'finite μ' $\approx$ 'finite ρ'



Thermodynamics

- Want to compute the most general thermodynamic quantity: grand potential
  - $\Omega = -pV = -T \ln Z$
  - $Z \equiv \mathrm{Tr}\,\rho \equiv \mathrm{Tr}\exp[-\beta\,(H-\mu N)] = \int \mathcal{D}\phi \exp\left\{-\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d x\ (\mathcal{L}-\mu N)\right\}$

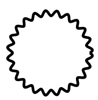
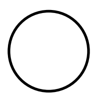
- Why?
  - In cosmology need the pressure at $\mu = 0$
  - For hydrodynamics in heavy ion collisions want p also for finite μ
  - At high μ/T quark stars, etc.
  - The problem interesting even as such

- Methods: lattice QCD and perturbation theory
  - Finite μ problem for lattice simulations due to complexity of action
  - PT requires in principle asymptotically high energies

Partition function in PT

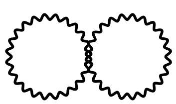
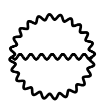
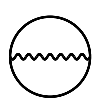
- Expand the functional integral in powers of g^2
  - Get vacuum diagrams with finite T Feynman rules
    - p_0 integrals replaced by discrete sums, etc.

- To leading order:

$p_1 =$  $+$ 

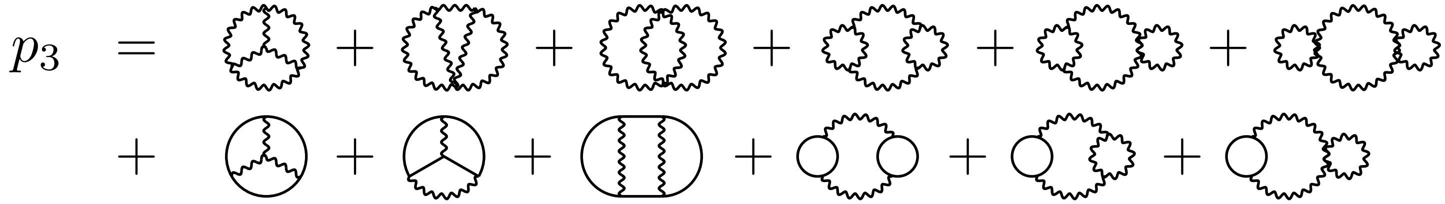
$$= \frac{\pi^2 T^4}{45}\left(N_c^2 - 1 + \frac{7N_c n_f}{4}\right) + \frac{N_c}{6} T^2 \sum_f \mu_f^2 + \frac{N_c}{12\pi^2}\sum_f \mu_f^4$$

- Two loops (Shuryak, Chin; 1978):

$p_2 =$  $+$  $+$ 

$$= -\frac{(N_c^2-1)g^2}{16}\left\{\frac{T^4}{9}\left(N_c + \frac{5n_f}{4}\right) + \frac{T^2}{2\pi^2}\sum_f \mu_f^2 + \frac{1}{4\pi^4}\sum_f \mu_f^4\right\}$$

- Three loops:



- Here run into trouble: uncancelled IR divergences from ring diagrams
  - Solution: (re)summation of IR sensitive terms to all orders
    $\Longrightarrow$ Get contributions nonanalytic in g^2
  - Have to treat the limits of high and low T/μ separately

High T: dimensional reduction

- At high T all non-zero Matsubara modes of fields get large effective masses
  - Integrate them out to get an effective 3d theory for the bosonic zero modes[1]
- Contributions of different momentum scales get separated: $p_{\text{QCD}} \equiv p_{\text{E}}+p_{\text{M}}+p_{\text{G}}$
  - p_{E}: Contribution of scale $2\pi T$ thorough strict perturbation expansion of p
    * $p_{\text{E}} \simeq 1 + g^2 + g^4 + \mathcal{O}(g^6)$
  - p_{M}: Scale gT through the pressure of 3d YM + adjoint Higgs (A_0) theory
    * $p_{\text{M}} \simeq g^3 + g^4(1+\ln g) + g^5 + g^6 \ln g + \mathcal{O}(g^6)$
  - p_{G}: Scale g^2T through the pressure of 3d YM theory
    * $p_{\text{G}} \simeq g^6 \ln g + \mathcal{O}(g^6)$
  - $\mathcal{O}(g^6 \ln g)$ requires extending 3d PT up to 4 loops (Y. Schröder)
  - Effective theories studied also on the lattice (K. Rummukainen, etc.)
- Done all the way at $\mu = 0$[2], but at finite μ[3] only to $\mathcal{O}(g^4 \ln g)$

[1] Ginsparg, Nucl. Phys. B 170 (1980) 388; Appelquist, Pisarski, Phys. Rev. D 23 (1981) 2305
[2] Kajantie, Laine, Rummukainen, Schröder, hep-ph/0211321
[3] Toimela, Phys. Lett. B **124** (1983) 407.

- Generalization of order $g^6 \ln g$ result to finite μ[4]:
  - Parameters of effective theories modified[5] by μ, new terms appear to $\mathcal{L}_{\text{eff}}$
  - Strict perturbation expansion of p only known to two loops at finite μ
    * Compute all 3-loop 1PI vacuum graphs in $3-2\epsilon$ dimensions at arbitrary T, μ (hard part)
    * Mostly same techniques work as at $\mu = 0$: Fourier-transform propagators into coordinate space using

$$\int \frac{\mathrm{d}^3 \mathrm{q}}{(2\pi)^3} \frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{\mathbf{q}^2 + (q_0 - i\mu)^2} = \frac{\exp[-(|q_0| - i\mu\,\mathrm{sign}\,q_0)\,r]}{4\pi T}$$

      and start computing

- (Mile-long) result expressible in terms of functions
  - $\aleph(n,z) \equiv \zeta'(-n,z) + (-1)^{n+1}\zeta'(-n,z^*),\ n = 0,1,2,3$
  - $\aleph(z) \equiv \Psi(z) + \Psi(z^*)$

  with $z \equiv 1/2 - i\mu/(2\pi T) \equiv 1/2 - i\bar{\mu}$

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[4] AV, hep-ph/0305183

[5] Hart, Laine, Philipsen, Nucl. Phys. B **586** (2000) 443 [hep-ph/0004060].

$$
\begin{aligned}
\frac{p_{\mathsf{QCD}}(T,\mu)}{T^4\Lambda^{-2\epsilon}} &= \frac{p_{\mathsf{E}}(T,\mu)+p_{\mathsf{M}}(T,\mu)+p_{\mathsf{G}}(T)}{T^4\Lambda^{-2\epsilon}} \\
&= g^0\Big\{\alpha_{\mathsf{E1}}\Big\} + g^2\Big\{\alpha_{\mathsf{E2}}\Big\} + \frac{g^3}{(4\pi)}\Big\{\frac{d_A}{3}\alpha_{\mathsf{E4}}^{3/2}\Big\} \\
&+ \frac{g^4}{(4\pi)^2}\Big\{\alpha_{\mathsf{E3}} - d_A C_A\Big[\alpha_{\mathsf{E4}}\Big(\frac{1}{4\epsilon}+\frac{3}{4}+\ln\frac{\bar\Lambda}{2gT\alpha_{\mathsf{E4}}^{1/2}}\Big)+\frac{1}{4}\alpha_{\mathsf{E5}}\Big]\Big\} \\
&+ \frac{g^5}{(4\pi)^3}\Big\{d_A\alpha_{\mathsf{E4}}^{1/2}\Big[\frac{1}{2}\alpha_{\mathsf{E6}} - C_A^2\Big(\frac{89}{24}+\frac{\pi^2}{6}-\frac{11}{6}\ln 2\Big)\Big]\Big\} \\
&+ \frac{g^6}{(4\pi)^4}\Big\{d_A C_A\Big(\alpha_{\mathsf{E6}}+\alpha_{\mathsf{E4}}\alpha_{\mathsf{E7}}\Big)\ln\Big[g\alpha_{\mathsf{E4}}^{1/2}\Big] + \frac{16}{3n_f^2}\Big(\sum_f \bar\mu\Big)^2 d_A D T_F^2 \ln\Big[g\alpha_{\mathsf{E4}}^{1/2}\Big] \\
&- 8\, d_A C_A^3\Big[\Big(\frac{43}{32}-\frac{491}{6144}\pi^2\Big)\ln\Big[g\alpha_{\mathsf{E4}}^{1/2}\Big] + \Big(\frac{43}{48}-\frac{157}{3072}\pi^2\Big)\ln\Big[gC_A^{1/2}\Big]\Big]\Big\} + \mathcal{O}(g^6), \\
\alpha_{\mathsf{E1}} &= \frac{\pi^2}{45}\,\frac{1}{n_f}\sum_f\Big\{d_A+\Big(\frac{7}{4}+30\bar\mu^2+60\bar\mu^4\Big)d_F\Big\}, \\
\alpha_{\mathsf{E2}} &= -\frac{d_A}{144}\,\frac{1}{n_f}\sum_f\Big\{C_A+\frac{T_F}{2}\Big(1+12\bar\mu^2\Big)\Big(5+12\bar\mu^2\Big)\Big\},
\end{aligned}
$$

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\begin{aligned}
\alpha_{\mathsf{E3}} &= \frac{d_A}{144}\Bigg[\frac{1}{n_f}\sum_f \Bigg\{ C_A^2\bigg(\frac{12}{\epsilon} + \frac{194}{3}\ln\frac{\bar{\Lambda}}{4\pi T} + \frac{116}{5} + 4\gamma - \frac{38}{3}\frac{\zeta'(-3)}{\zeta(-3)} + \frac{220}{3}\frac{\zeta'(-1)}{\zeta(-1)}\bigg) \\
&+ C_A T_F\bigg(12\left(1+12\bar{\mu}^2\right)\frac{1}{\epsilon} + \left(\frac{169}{3} + 600\bar{\mu}^2 - 528\bar{\mu}^4\right)\ln\frac{\bar{\Lambda}}{4\pi T} + \frac{1121}{60} + 8\gamma \\
&+ 2\left(127+48\gamma\right)\bar{\mu}^2 - 644\bar{\mu}^4 + \frac{268}{15}\frac{\zeta'(-3)}{\zeta(-3)} + \frac{4}{3}\left(11+156\bar{\mu}^2\right)\frac{\zeta'(-1)}{\zeta(-1)} \\
&+ 24\Big[52\,\aleph(3,z) + 144 i\bar{\mu}\,\aleph(2,z) + \left(17-92\bar{\mu}^2\right)\aleph(1,z) + 4i\bar{\mu}\,\aleph(0,z)\Big]\bigg) \\
&+ C_F T_F\bigg(\frac{3}{4}\left(1+4\bar{\mu}^2\right)\left(35+332\bar{\mu}^2\right) - 24\left(1-4\bar{\mu}^2\right)\frac{\zeta'(-1)}{\zeta(-1)} \\
&- 144\Big[12 i\bar{\mu}\,\aleph(2,z) - 2\left(1+8\bar{\mu}^2\right)\aleph(1,z) - i\bar{\mu}\left(1+4\bar{\mu}^2\right)\aleph(0,z)\Big]\bigg) \\
&+ T_F^2\bigg(\frac{4}{3}\left(1+12\bar{\mu}^2\right)\left(5+12\bar{\mu}^2\right)\ln\frac{\bar{\Lambda}}{4\pi T} + \frac{1}{3} + 4\gamma + 8\left(7+12\gamma\right)\bar{\mu}^2 + 112\bar{\mu}^4 - \frac{64}{15}\frac{\zeta'(-3)}{\zeta(-3)} \\
&- \frac{32}{3}\left(1+12\bar{\mu}^2\right)\frac{\zeta'(-1)}{\zeta(-1)} - 96\Big[8\,\aleph(3,z) + 12 i\bar{\mu}\,\aleph(2,z) - 2\left(1+2\bar{\mu}^2\right)\aleph(1,z) - i\bar{\mu}\,\aleph(0,z)\Big]\bigg)\Bigg\} \\
&+ 288\,T_F^2\frac{1}{n_f^2}\sum_{f\,g}\Bigg\{2\left(1+\gamma\right)\bar{\mu}_f^2\bar{\mu}_g^2 - \Big[\aleph(3,z_f+z_g) + \aleph(3,z_f+z_g^*) \\
&+ 4i\bar{\mu}_f\Big(\aleph(2,z_f+z_g) + \aleph(2,z_f+z_g^*)\Big) - 4\bar{\mu}_g^2\,\aleph(1,z_f) - \left(\bar{\mu}_f+\bar{\mu}_g\right)^2\aleph(1,z_f+z_g) \\
&- \left(\bar{\mu}_f-\bar{\mu}_g\right)^2\aleph(1,z_f+z_g^*) - 4i\bar{\mu}_f\bar{\mu}_g^2\,\aleph(0,z_f)\Big]\Bigg\}\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\alpha_{\mathsf{E4}} &= \frac{1}{3}\,\frac{1}{n_f}\sum_f \left\{ C_A + T_F \left(1 + 12\bar{\mu}^2\right) \right\}, \\
\alpha_{\mathsf{E5}} &= \frac{1}{3}\,\frac{1}{n_f}\sum_f \left\{ 2\,C_A\left(\ln\frac{\bar{\Lambda}}{4\pi T} + \frac{\zeta'(-1)}{\zeta(-1)}\right) + T_F\left(\left(1 + 12\bar{\mu}^2\right) \left(2\ln\frac{\bar{\Lambda}}{4\pi T} + 1\right) + 24\,\aleph(1,z)\right)\right\}, \\
\alpha_{\mathsf{E6}} &= \frac{1}{9}\,\frac{1}{n_f}\sum_f \left\{ C_A^2\left(22\ln\frac{e^{\gamma}\bar{\Lambda}}{4\pi T} + 5\right) + C_A T_F\left(2\left(7 + 132\bar{\mu}^2\right)\ln\frac{e^{\gamma}\bar{\Lambda}}{4\pi T} + 9 + 132\bar{\mu}^2 + 8\gamma + 4\,\aleph(z)\right)\right. \\
&- \left. 18\,C_F T_F\left(1 + 12\bar{\mu}^2\right) - 4\,T_F^2\left(1 + 12\bar{\mu}^2\right)\left(2\ln\frac{\bar{\Lambda}}{4\pi T} - 1 - \aleph(z)\right)\right\}, \\
\alpha_{\mathsf{E7}} &= \frac{1}{3}\,\frac{1}{n_f}\sum_f \left\{ C_A\left(22\ln\frac{e^{\gamma}\bar{\Lambda}}{4\pi T} + 1\right) - 4\,T_F\left(2\ln\frac{\bar{\Lambda}}{4\pi T} - \aleph(z)\right)\right\}.
\end{aligned}
$$

Lattice tests

- No experimental data for p available $\Rightarrow$ lattice simulations[6] [7] [8] provide the only independent check for the results

- Finite μ and very high T problematic on the lattice
  - Study p at small μ and quark number susceptibilities[9] at $\mu = 0$
    * $\chi_{ijk\ldots} \equiv \frac{\partial^n p}{\partial\mu_i \partial\mu_j \partial\mu_k \ldots}$
  - Keep the temperature sufficiently low ($T \leq 5\,T_c$); expect deviations there

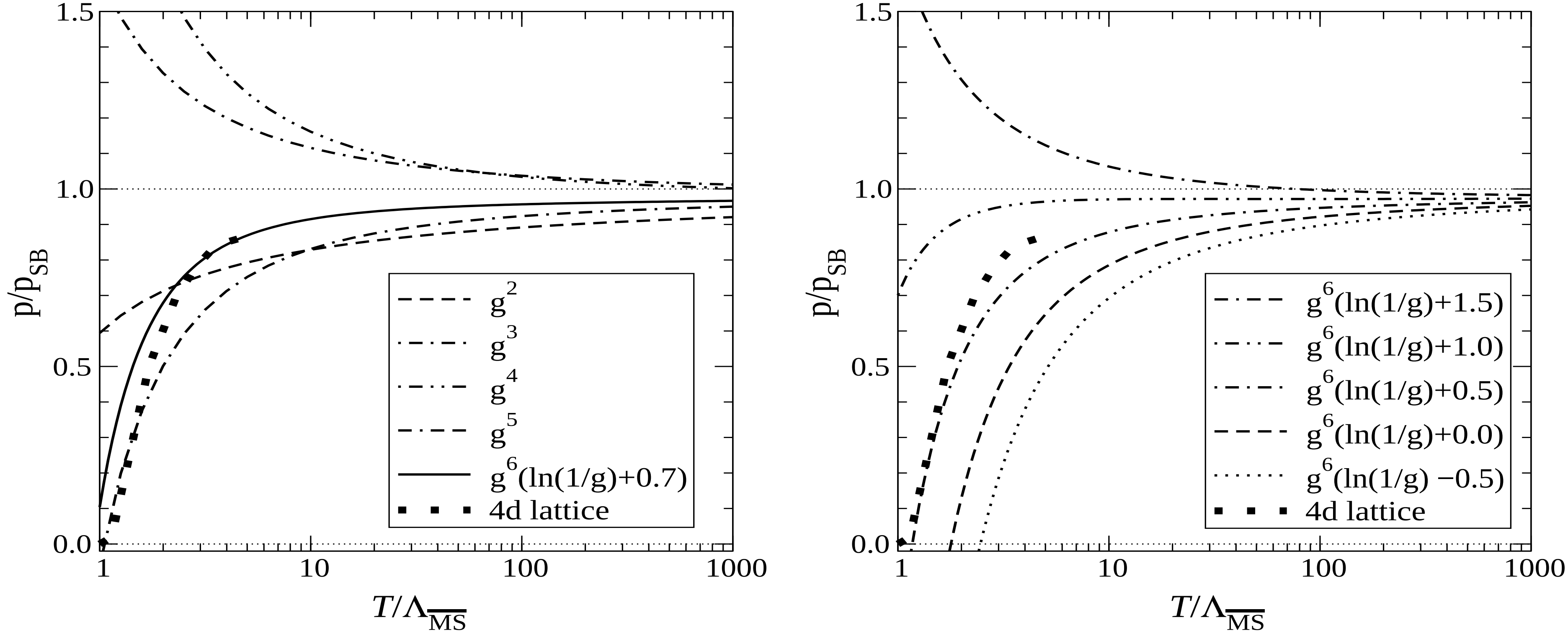
- Most lattice studies use quenched QCD with two light flavors $\Rightarrow$ in PT expressions set explicit factors of n_f to zero

[6] Gavai, Gupta, hep-lat/0211015; hep-lat/0303013

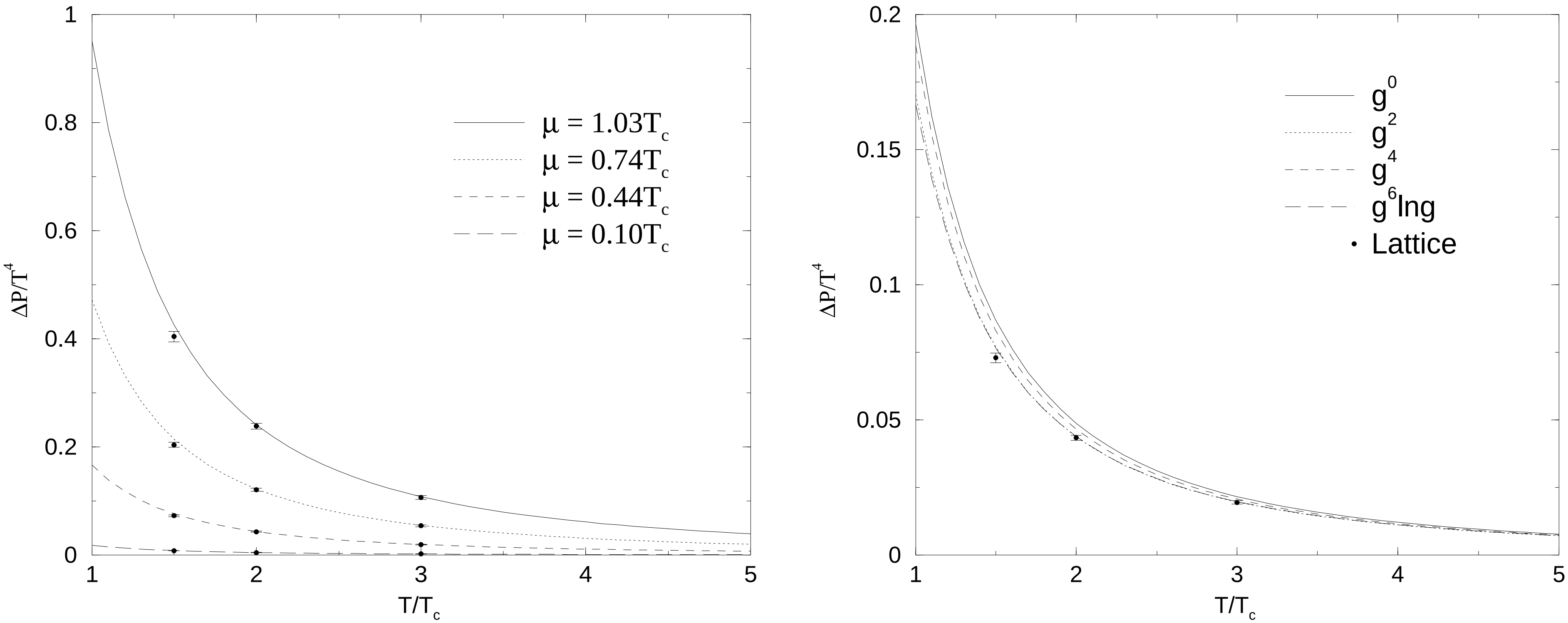
[7] Gavai, Gupta, Majumdar, Phys. Rev. D **65** (2002) 054506 [hep-lat/0110032]

[8] Bernard *et al.* [MILC Collaboration], [hep-lat/0209079]

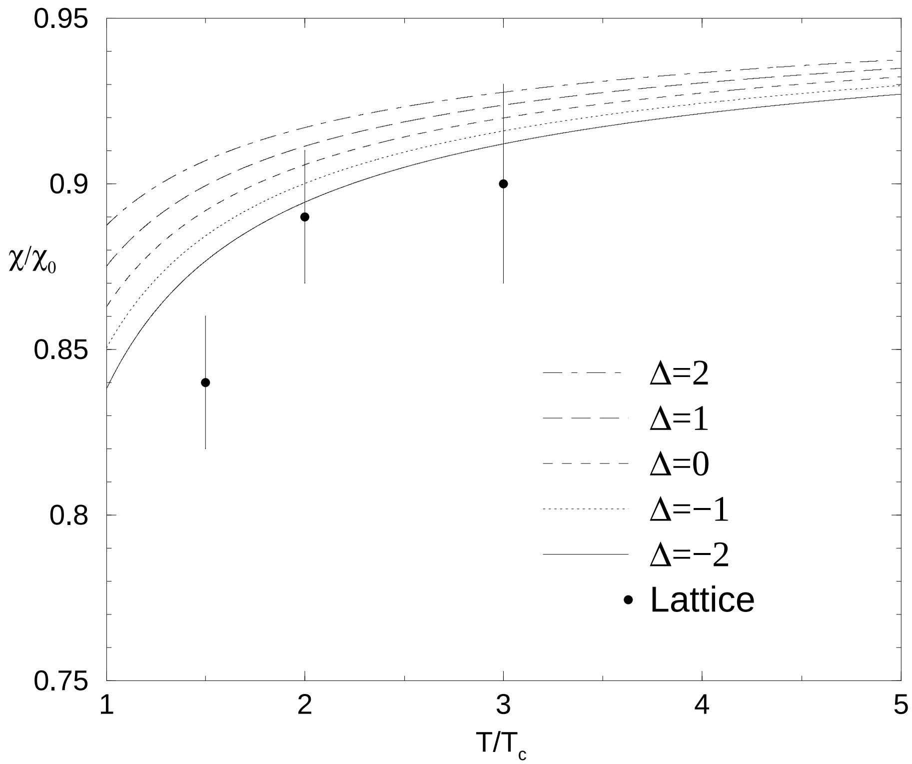
[9] AV, Phys. Rev. D **67** (2003) 074032 [hep-ph/0212283].

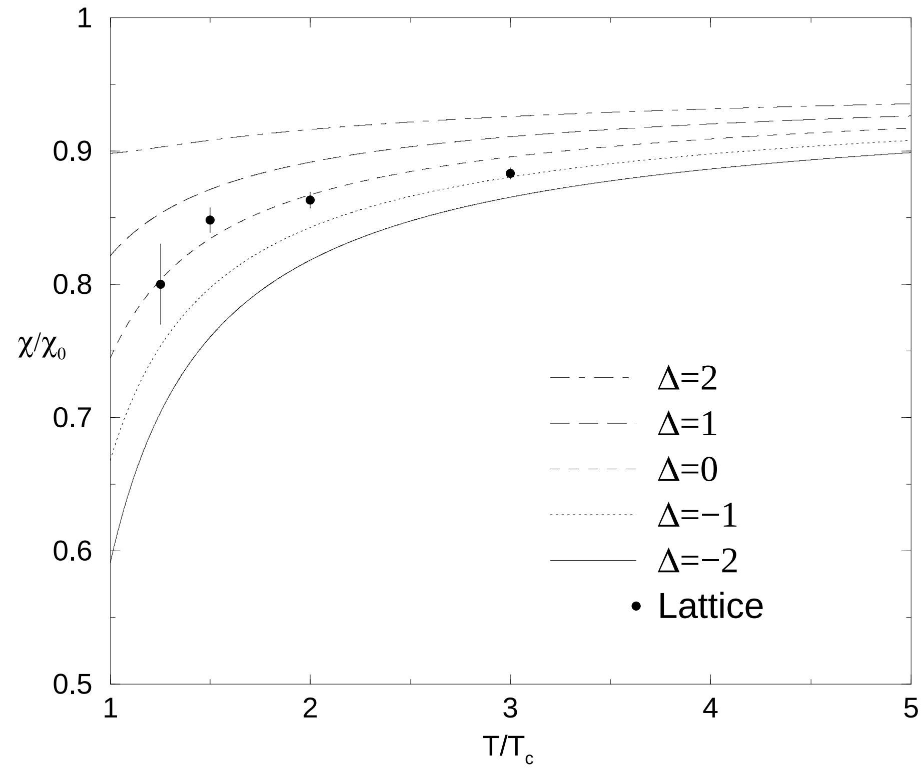


The perturbative expansion of $p(T, \mu = 0)$ from [2] plotted against 4d lattice data of G. Boyd *et al.*, hep-lat/9602007. Here $n_f = 0$, $N_c = 3$.

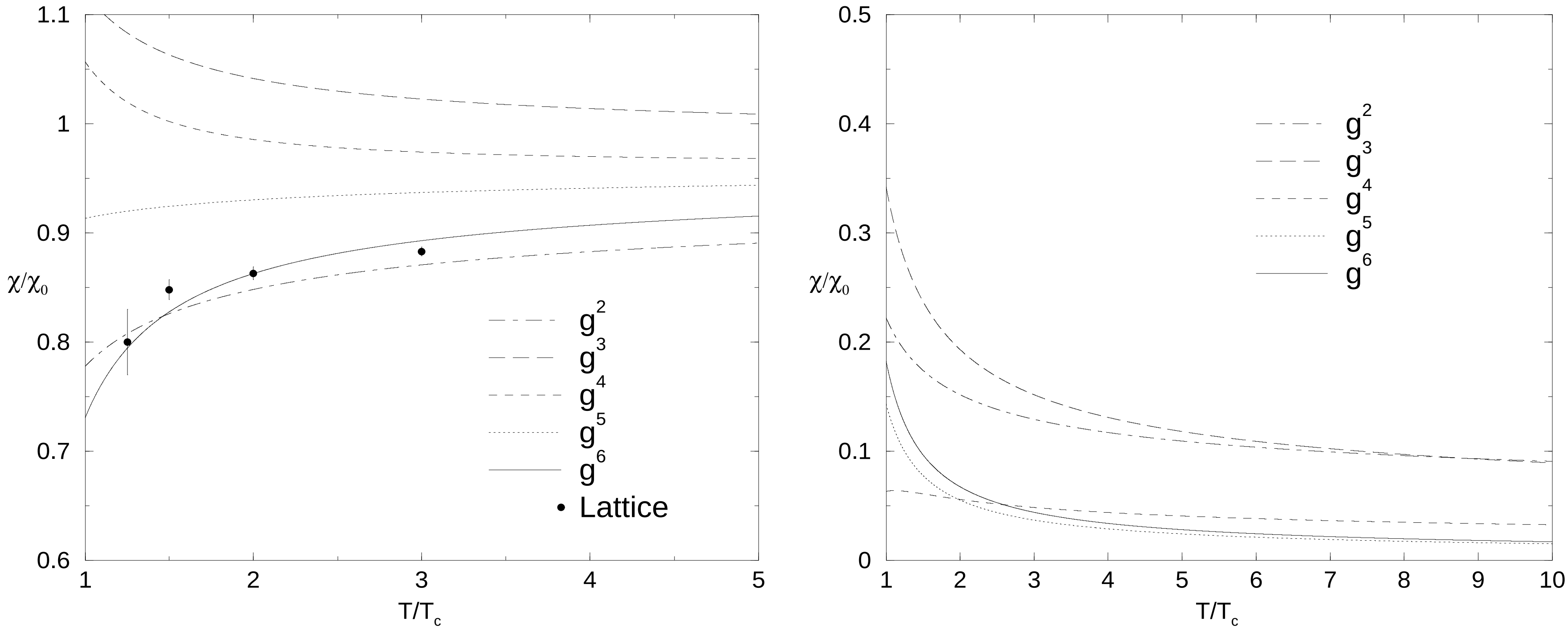


The perturbative [4] and lattice [6] results for $\Delta P \equiv p(T,\mu) - p(T,\mu=0)$. Again $n_f = 0$, $N_c = 3$.





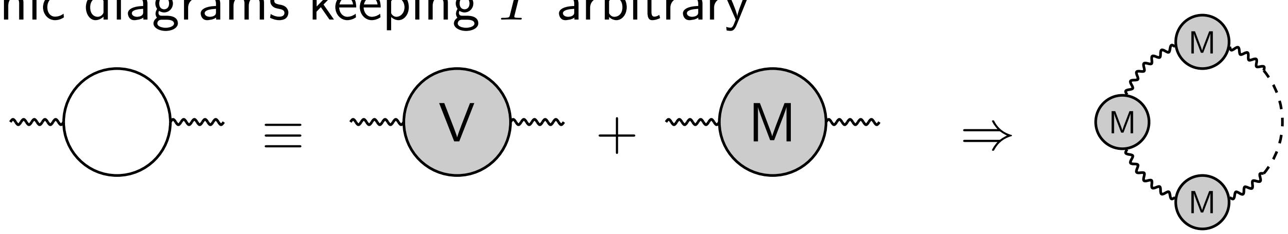
χ_{uu} at $n_f = 0$ (left) and $n_f = 2$ (right). PT result from [9] and lattice data from [6,7]. The ideal gas result $\chi_0 = T^2$.



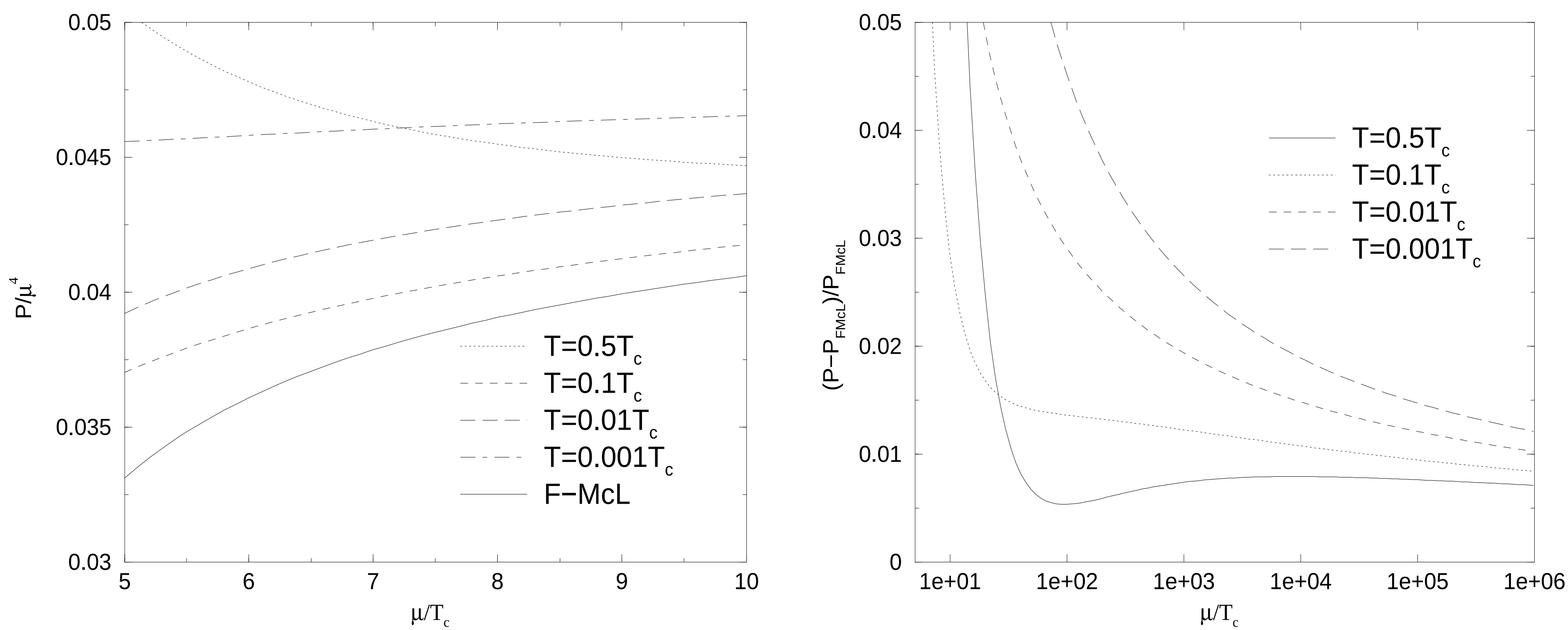
χ_{uu} at $n_f = 2$ [9] and plotted at different perturbative orders. Lattice data again from [7].

Low T: summation of ring diagrams

- Even at low T PT works, if μ large enough
  - Can never produce CSC, CFL or any other non-perturbative physics
  - No DR theory available $\Rightarrow$ need to explicitly sum over all ring diagrams $\Rightarrow$ non-analytic contributions starting at $\mathcal{O}(g^4 \ln g)$
  - At $T = 0$ done by Freedman and McLerran (1977) to $\mathcal{O}(g^4)$ and the result analytically verified in [5]
    - $4.24 \pm 0.12 \; = \; 17/4$
  - The limit of small but non-zero T the most problematic; need to sum also bosonic diagrams keeping T arbitrary



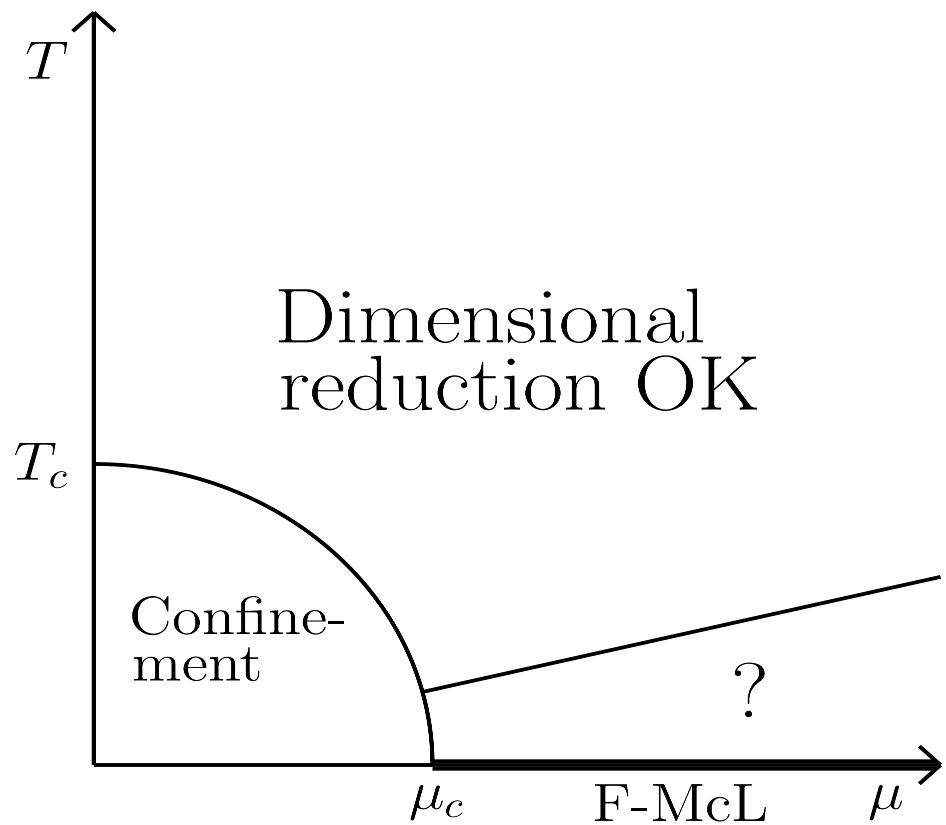
- DR breaks down at low T and high μ [4,5]; for it to be well-defined, need $\mu_f \lesssim 4T$
  - Try anyway matching the DR result to the $T = 0$ one at low T



The DR expression for p [5] compared with the $T = 0$ result as a function of $\mu_u = \mu_d \equiv \mu$.

What next?

- The PT result [5] for the pressure covers now most of the μ-T plane
  - Even the condition $\mu \lesssim 4T$ for DR can perhaps be loosened
- The limit $\mu/T \gg 1$ must definitely be improved
- At $\mu = 0$ need the $\mathcal{O}(g^6)$ term giving first non-pert. contributions



Conclusions

- Knowing the grand potential, or the pressure, of QCD on the whole μ-T plane is obviously a goal worth fighting for

- Lattice QCD the only fundamentally non-perturbative tool available - however, its use is quite limited at present

- Perturbation theory has given good results at high temperatures; especially the chemical potential dependent part of the pressure produced with good accuracy

- The limit of small T still problematic, but not for long!