Fun with thermal dimension-six operators[1,2]

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[2] Based on collaboration with Philipp Schicho and York Schröder.

Context

Dimensionally reduced effective theory for hot QCD

$$
\begin{aligned}
L_{\rm EQCD} \quad \equiv \quad & \frac{1}{4} F^a_{ij} F^a_{ij} + \frac{1}{2} \mathcal{D}^{ab}_i A^b_0 \, \mathcal{D}^{ac}_i A^c_0 + \frac{m_{\rm E}^2}{2} A^a_0 A^a_0 \\
+ \quad & \frac{\lambda_{\rm E}}{4} X^{abcd} A^a_0 A^b_0 A^c_0 A^d_0 + \frac{\kappa_{\rm E}}{4} A^a_0 A^a_0 A^b_0 A^b_0 \, .
\end{aligned}
$$

Developed for studying high-temperature thermodynamics.[3]

Remarkably,[4] also applies to soft light-cone observables.[5]

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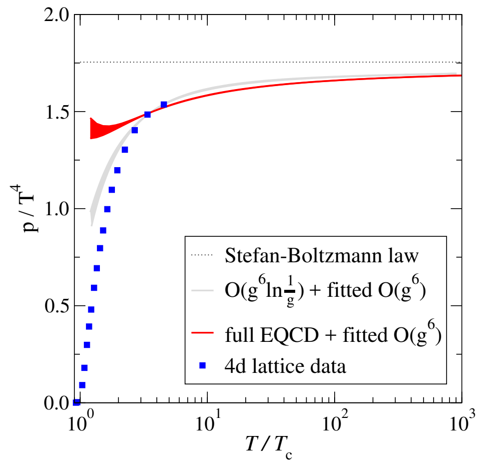
[3] P.H. Ginsparg, *First and second order phase transitions in gauge theories at finite temperature*, NPB 170 (1980) 388; T. Appelquist and R.D. Pisarski, *High-temperature Yang-Mills theories and three-dimensional Quantum Chromodynamics*, PRD 23 (1981) 2305.

[4] S. Caron-Huot, *$O(g)$ plasma effects in jet quenching*, 0811.1603.

[5] e.g. M. Panero, K. Rummukainen and A. Schäfer, *Lattice Study of the Jet Quenching Parameter*, 1307.5850; J. Ghiglieri *et al*, *Next-to-leading order thermal photon production in a weakly coupled quark-gluon plasma*, 1302.5970.

However the EQCD description fails close to T_c

Pure-glue transition is driven by Z(3) symmetry,[6] which is not explicit in EQCD. There is partial dynamical re-generation,[7] however in practice non-perturbative EQCD does **not** work well.[8]



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[6] B. Svetitsky and L.G. Yaffe, *Critical Behavior at Finite Temperature Confinement Transitions*, NPB 210 (1982) 423.

[7] K. Kajantie *et al*, *Phase diagram of 3d SU(3) + adjoint Higgs theory*, hep-lat/9811004.

[8] A. Hietanen *et al*, *Three-dimensional physics and the pressure of hot QCD*, 0811.4664.

Terminology

"hard scale" $\sim \pi T$:

integrated out from QCD in order to arrive at EQCD.

"soft scale" $\sim m_\mathrm{E} \sim gT$, where $g \equiv \sqrt{4\pi\alpha_\mathrm{s}}$:

mass/momentum scale of the EQCD field A_0^a.

"ultrasoft scale" $\sim g^2T/\pi$:

momentum scale of MQCD, obtained by integrating out A_0^a.

Conceptual clarifications

Often pQCD is said to fail at any reasonable temperature, because of **large soft corrections**, e.g. $|\text{NLO} - \text{LO}| \gtrsim \text{LO}$.

However, these effects can in principle be studied non-perturbatively through EQCD.

More troublesome is poor convergence at the **hard scale**, which is responsible for the breakdown of EQCD.

3-loop hard correction to $g_{\rm E}^2$ $(= g^2/[\mathcal{Z}_B + \delta\mathcal{Z}_B])$:[9]

$$
\begin{aligned}
\Gamma^{(2)}_{\rm EQCD}[B] &= \frac{1}{2} B^a_i(q)\, B^a_j(-q)\; (q^2\delta_{ij} - q_i q_j)\,(\mathcal{Z}_B + \delta\mathcal{Z}_B)\;, \\
\mathcal{Z}_B &= 1 - \frac{g^2 N_{\rm c}}{(4\pi)^2}\Big[\frac{22L}{3} + \frac{1}{3}\Big] - \frac{g^4 N_{\rm c}^2}{(4\pi)^4}\Big[\frac{68L}{3} + \frac{341}{18} - \frac{10\zeta_3}{9}\Big] \\
&\quad - \frac{g^6 N_{\rm c}^3}{(4\pi)^6}\Big[\frac{748L^2}{9} + \Big(\frac{6608}{27} - \frac{10982\zeta_3}{135}\Big)L + (\text{finite})\Big] + \mathcal{O}(g^8)\;, \\
\delta\mathcal{Z}_B &= \frac{g^6 N_{\rm c}^3}{(4\pi)^6}\frac{61\zeta_3}{5\epsilon} + \mathcal{O}(g^8)\;, \quad L \equiv \ln\Big(\frac{\bar\mu e^{\gamma_{\rm E}}}{4\pi T}\Big)\;.
\end{aligned}
$$

What is the IR divergence?? (No counterpart in EQCD!)

[9] I. Ghisoiu, *Three-loop Debye mass and effective coupling in thermal QCD*, PhD thesis (2013) [https://pub.uni-bielefeld.de/publication/2632705]; I. Ghisoiu and Y. Schröder, poster at SEWM14 [http://www.sewm14.unibe.ch/ghisoiu.pdf].

Dimension-six operators

Chapman action:[10]

$$
\begin{aligned}
\delta L_{\rm EQCD} \quad = \quad & \SumInt_P' \frac{2g_{\rm E}^2}{P^6}\,{\rm tr}\Big\{c_1\,(D_\mu F_{\mu\nu})^2 + c_2\,(D_\mu F_{\mu 0})^2 \\
+ \quad & i g_{\rm E}\,[c_3\,F_{\mu\nu}F_{\nu\rho}F_{\rho\mu} + c_4\,F_{0\mu}F_{\mu\nu}F_{\nu 0} + c_5\,A_0(D_\mu F_{\mu\nu})F_{0\nu}] \\
+ \quad & g_{\rm E}^2\,[c_6\,A_0^2 F_{\mu\nu}^2 + c_7\,A_0 F_{\mu\nu} A_0 F_{\mu\nu} \\
& \qquad + c_8\,A_0^2 F_{0\mu}^2 + c_9\,A_0 F_{0\mu} A_0 F_{0\mu}] \\
+ \quad & g_{\rm E}^4\,[c_{10} A_0^6\,]\Big\}\;.
\end{aligned}
$$

(This basis is slightly redundant, which offers for nice crosschecks.)

[10] S. Chapman, *A New dimensionally reduced effective action for QCD at high temperature*, hep-ph/9407313.

Chapman coefficients

Because we are going to do loops with the Chapman operators, their coefficients are needed in $d = 3 - 2\epsilon$ dimensions. It is convenient to employ the background field gauge.[11]

A rather optimal way to determine the coefficients is from the 5-point function (20 independent colour/Lorentz structures).

(We also computed the 2-, 3- and 6-point functions.)

[11] L.F. Abbott, *The Background Field Method Beyond One Loop*, NPB 185 (1981) 189.

Leading-order results ($d = 3 - 2\epsilon$)

$$c_1 = \frac{41-d}{120}\,, \quad c_2 = \frac{(d-1)(d-5)}{120}\,, \quad c_3 = \frac{1-d}{180}\,,$$

$$c_4 - 2c_7 = \frac{(41-d)(5-d)}{60}\,, \quad c_5 - 2c_7 = \frac{(21-d)(5-d)}{30}\,,$$

$$c_6 + c_7 = \frac{(d-25)(5-d)}{24}\,, \quad c_8 = \frac{(5-d)(3-d)(d-1)}{20}\,,$$

$$c_9 = \frac{(5-d)(3-d)(d-1)}{30}\,, \quad c_{10} = \frac{(5-d)(3-d)(d-1)^2}{180}\,.$$

Operators coupling to c_8, c_9, c_{10} are "evanescent".

Loop effects from dimension-six operators

The goal now is to integrate out the scale m_{E}

In other words we reduce EQCD into MQCD:

$$L_{\mathrm{MQCD}} \equiv \frac{1}{4} F^a_{ij} F^a_{ij} \,.$$

We determine the coupling g_{M}^2 from the 2-point correlator in the background field gauge:

$$\delta\Gamma^{(2)}_{\mathrm{MQCD}}[B] = \frac{1}{2} B^a_i(q) B^a_j(-q)\,(q^2\delta_{ij} - q_i q_j)(Z_B + \delta Z_B)\,,$$

$$g^2_{\mathsf{M}} = \frac{g^2_{\mathsf{E}}}{Z_B + \delta Z_B}\,.$$

(We keep g^2_{E}, g^2_{M} dimensionless, showing T explicitly.)

1-loop level (blobs stand for Chapman vertices)

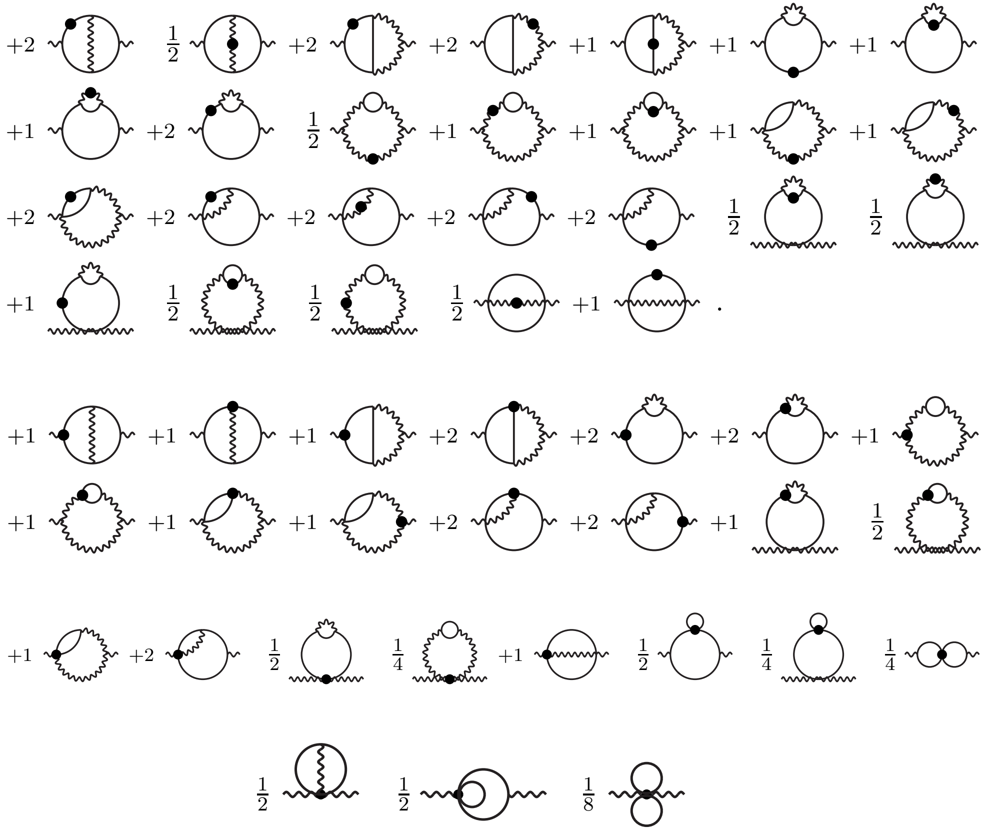
$$\delta\Gamma^{(2)}_{\rm MQCD}[B] \;=\; B^a_i(q)B^a_j(-q)\,(q^2\delta_{ij}-q_iq_j)\,\mathop{\Sigma\!\!\!\!\!\!\int}\nolimits'_P \frac{g_{\rm E}^4 N_{\rm c}^2}{P^6}\, I(m_{\rm E})$$

$$\times \quad \left\{ \frac{(4-d)(d-2)}{12}(c_1+c_2) + 3c_3 + (c_4 - 2c_7) + 4(c_6+c_7) \right\} \;,$$

$$I(m_{\rm E}) \;\equiv\; \int_p \frac{T}{p^2+m_{\rm E}^2} \;=\; -\frac{m_{\rm E} T \mu^{-2\epsilon}}{4\pi}\left[1+\mathcal{O}(\epsilon)\right] \;.$$

This is a finite contribution of $\mathcal{O}(g^4 m_{\rm E}/T) \sim \mathcal{O}(g^5)$.

2-loop diagrams



Summary: 1-loop and 2-loop results

$$Z_B = 1 + \Big(\frac{g_{\mathsf{E}}^2 N_c}{16\pi^2}\Big)^2 \frac{m_{\mathsf{E}}}{2\pi T}\Big(\frac{875\zeta_3}{72}\Big)$$

$$- \Big(\frac{g_{\mathsf{E}}^2 N_c}{16\pi^2}\Big)^3 \Big(\frac{1097\zeta_3}{549}\Big)\frac{61}{5}\Big\{L + 2\ln\Big(\frac{\bar{\mu}}{2m_{\mathsf{E}}}\Big) + \frac{\zeta_3'}{\zeta_3} - \gamma_{\mathsf{E}} + \frac{103771}{52656}\Big\} \,,$$

$$\delta Z_B = -\frac{g_{\mathsf{E}}^6 N_c^3}{(4\pi)^6}\Big(\frac{1097\zeta_3}{1098}\Big)\frac{61}{5\epsilon} \,.$$

This cancels $\frac{1097}{1098}$ of the IR divergence from hard scales!

Intermediate summary

After integrating out the scales $\sim \pi T$ and $\sim m_{\rm E}$, an IR divergence remains in the effective gauge coupling.

This remaining "1/1098" can be expressed as $(1098 = 61 \times 18)$

$$\delta \mathcal{Z}_B + \delta Z_B \;=\; \frac{g^6 N_{\rm c}^3 T^2}{(8\pi)^2} \left(\sum\!\!\!\!\!\!\int_P^{\prime} \frac{1}{P^6} \right) \frac{1}{45\epsilon} + \mathcal{O}(g^8) \; .$$

Contribution from MQCD?

$$\delta L_{\mathrm{MQCD}} = \SumInt_P' \frac{2g_{\mathrm{M}}^2}{P^6} \mathrm{tr}\{c_1 \,(D_i F_{ij})^2 + i g_{\mathrm{M}} c_3 \, F_{ij} F_{jk} F_{ki}\} \ .$$

To compute the UV divergence from this sector, we “shield” the IR. The method is “unphysical”, but sufficient for our purposes.

$$\langle A_k^a(p) A_l^b(q) \rangle \equiv \frac{\delta^{ab}\delta(p+q)}{p^2+m_{\mathrm{G}}^2}\left(\delta_{kl} - \frac{\alpha\, p_k p_l}{p^2+m_{\mathrm{G}}^2}\right) ,$$

$$\langle c^a(p) \bar{c}^b(q) \rangle \equiv \frac{\delta^{ab}\delta(p-q)}{p^2+m_{\mathrm{G}}^2} .$$

Divergence from 2-loop graphs

Apart from many gauge-dependent finite contributions, there is also a gauge-independent divergent contribution:

$$\begin{aligned}\delta\Gamma^{(2)}_{\rm IR}[B] &= \frac{1}{2}B^a_i(q)\,B^a_j(-q)\,(q^2\delta_{ij}-q_iq_j) \\ &\times \left(\sum\!\!\!\!\!\!\!\int_P{}' \frac{g_{\rm M}^6 N_{\rm c}^3}{P^6}\right)\frac{T^2 c_3}{(4\pi)^2 2\epsilon} \\ &= \frac{1}{2}B^a_i(q)\,B^a_j(-q)\,(q^2\delta_{ij}-q_iq_j) \\ &\times \frac{g^6 N_{\rm c}^3 T^2}{(8\pi)^2}\left(\sum\!\!\!\!\!\!\!\int_P{}' \frac{1}{P^6}\right)\left(-\frac{1}{45\epsilon}\right)+\mathcal{O}(g^7)\,.\end{aligned}$$

The divergence cancels perfectly, and all is fine, provided that dimension-six operators are included in EQCD and MQCD :)

Purely soft effects
(= another 3-loop computation)

Now integrate out $m_{\rm E}$ *without* Chapman vertices

To distinguish these effects from the previous ones, we denote the 2-point contribution by $\widetilde{Z}_B + \delta \widetilde{Z}_B$.

Direct computation up to 2 loops:[12,13]

$$\widetilde{Z}_B = 1 + \frac{g_{\rm E}^2 N_{\rm c} T}{48\pi m_{\rm E}} + \left(\frac{g_{\rm E}^2 N_{\rm c} T}{16\pi m_{\rm E}} \right)^2 \left(\frac{19}{18} + \frac{4\lambda}{3} \right) .$$

[12] P. Giovannangeli, *Two loop renormalization of the magnetic coupling in hot QCD*, hep-ph/0312307.

[13] $\lambda \equiv \frac{5\lambda_{\rm E} N_{\rm c}}{4 g_{\rm E}^2} + \frac{\kappa_{\rm E}(N_{\rm c}^2+1)}{2 g_{\rm E}^2 N_{\rm c}}$.

3-loop computation of $\widetilde{Z}_B + \delta\widetilde{Z}_B$

We make use of standard automated tools: QGRAF for diagram generation, integration-by-part identities (IBP) for systematic reduction to masters, FORM for efficient implementation.[14]

The master integrals are all known in analytic form, e.g.[15]

$$
\begin{aligned}
B_2 &\equiv \int_{p,q,r} \frac{1}{(p^2+m^2)(q^2+m^2)(p+r)^2(q+r)^2} \\
&= -\frac{m\mu^{-6\epsilon}}{(4\pi)^3}\Big(\frac{\bar{\mu}}{2m}\Big)^{6\epsilon}\Big\{\frac{1}{2\epsilon}+4+\epsilon\Big[26+\frac{25\zeta_2}{4}\Big]+\mathcal{O}(\epsilon^2)\Big\}\,.
\end{aligned}
$$

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[14] P. Nogueira, *Automatic Feynman graph generation*, J. Comput. Phys. 105 (1993) 279; S. Laporta, *High precision calculation of multiloop Feynman integrals by difference equations*, hep-ph/0102033; A. von Manteuffel and C. Studerus, *Reduze 2 - Distributed Feynman Integral Reduction*, 1201.4330; J. Kuipers, T. Ueda, J.A.M. Vermaseren and J. Vollinga, *FORM version 4.0*, 1203.6543.

[15] A.K. Rajantie, *Feynman diagrams to three loops in 3d field theory*, hep-ph/9606216.

3-loop result after renormalization of $m_{\rm E}^2$:[16]

$$
\begin{aligned}
\widetilde{Z}_B^{(3)} + \delta\widetilde{Z}_B^{(3)} &= \left(\frac{g_{\mathsf{E}}^2 N_{\rm c} T}{16\pi m_{\mathsf{E}}}\right)^3 \Bigg\{ \frac{1}{6\epsilon} + \left[1 + \frac{8(\kappa_2 - 4\lambda)}{3}\right] \ln\left(\frac{\bar{\mu}}{2m_{\mathsf{E}}}\right) \\
&+ \frac{2(23510 - 12600\zeta_2 - 1101\ln 2)}{945} \\
&+ \frac{52\lambda + 24\lambda^2 - \kappa_1(5 - 8\ln 2) + \kappa_2(19 - 24\ln 2)}{9} \Bigg\} \,.
\end{aligned}
$$

Again there is an uncancelled IR divergence.

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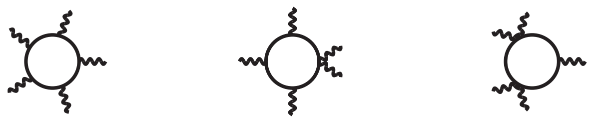
16 $\kappa_1 \equiv \frac{\lambda_{\mathsf{E}}(N_{\rm c}^2+36)}{2g_{\mathsf{E}}^2 N_{\rm c}} + \frac{10\kappa_{\mathsf{E}}}{g_{\mathsf{E}}^2 N_{\rm c}}$, $\kappa_2 \equiv \frac{\lambda_{\mathsf{E}}^2(N_{\rm c}^2+36)}{4g_{\mathsf{E}}^4} + \frac{10\lambda_{\mathsf{E}}\kappa_{\mathsf{E}}}{g_{\mathsf{E}}^4} + \frac{2\kappa_{\mathsf{E}}^2(N_{\rm c}^2+1)}{g_{\mathsf{E}}^4 N_{\rm c}^2}$.

MQCD contribution

Now we consider Chapman operators induce by the scale $m_{\rm E}$.

$$\delta L_{\rm MQCD} = \frac{g_{\rm M}^2 T}{16\pi m_{\rm E}^3} \, {\rm tr}\{\tilde{c}_1 \, (D_i F_{ij})^2 + i g_{\rm M} \tilde{c}_3 \, F_{ij} F_{jk} F_{ki}\} \; .$$



The coefficients read[17]

$$\tilde{c}_1 \; = \; -\frac{1}{120} \, , \quad \tilde{c}_3 \; = \; -\frac{1}{180} \; .$$

[17] Originally: P. Giovannangeli (unpublished, 2005); C.P. Korthals Altes, *The unbearable smallness of magnetostatic QCD corrections*, 1801.00019.

2-loop result from Chapman operators

The computation is like before, just with different coefficients.

$$\delta\widetilde{\Gamma}^{(2)}_{\rm IR}[B] = \frac{1}{2}B_i^a(q)\,B_j^a(-q)\,(q^2\delta_{ij} - q_iq_j) \times \left(\frac{g_{\rm E}^2 N_{\rm c} T}{16\pi m_{\rm E}}\right)^3 \left\{-\frac{1}{45\epsilon} + \text{(finite)}\right\} .$$

- divergences on both sides are gauge independent.
- all crosschecks we could think of have passed.
- but there is no cancellation :(

A possible interpretation

EQCD is a confining theory, so physics at the scale m_E^2 may be affected by non-perturbative ambiguities of $\mathcal{O}(g^4T^2/\pi^2)$.

This is clear for "physical states" (i.e. screening lengths)[18], but perhaps also for the IR-sensitivity of Lagrangian parameters?

Inserting an ambiguity inside the 1-loop result yields

$$\frac{g_\mathrm{E}^2 N_\mathrm{c} T}{48\pi\left[m_\mathrm{E}^2 + \frac{\beta}{\epsilon_\mathrm{IR}}\left(\frac{g_\mathrm{E}^2 N_\mathrm{c} T}{16\pi}\right)^2\right]^{1/2}} \leftrightarrow -\frac{\beta}{6\epsilon_\mathrm{IR}}\left(\frac{g_\mathrm{E}^2 N_\mathrm{c} T}{16\pi m_\mathrm{E}}\right)^3 , \quad \beta = -\frac{13}{15} .$$

[18] A.K. Rebhan, *Non-Abelian Debye mass at next-to-leading order*, hep-ph/9308232; P.B. Arnold and L.G. Yaffe, *The non-Abelian Debye screening length beyond leading order*, hep-ph/9508280.

Conclusions

Even if the Debye scale $m_{\rm E} \sim gT$ is formally larger than the magnetic scale $\sim \frac{g^2T}{\pi}$, it plays an essential role in IR dynamics.

In terms of an IR divergence in the 3-loop gauge coupling, it is 1097 times more important than the magnetic scale.

Chapman operators are needed in EQCD for good precision, and are a likely culprit for its failure close to $T_{\rm c}$.

We also find "trouble" if we integrate out the Debye scale with high precision: once Chapman operators are included in MQCD, $g^2_{\rm M}$ needs to be simultaneously determined non-perturbatively.