Going with the flow

a solution to your sign problems

Gökçe Başar

University of Illinois, Chicago

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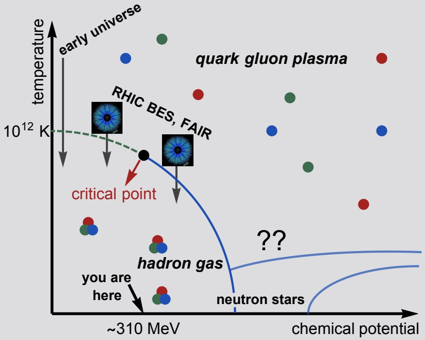
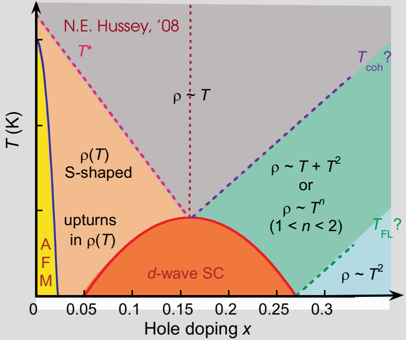
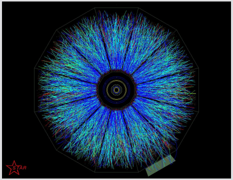
with A. Alexandru, P. Bedaque, G. Ridgway, N. Warrington

Motivations

first-principles studies of strongly interacting systems
(equation of state, transport, etc...)



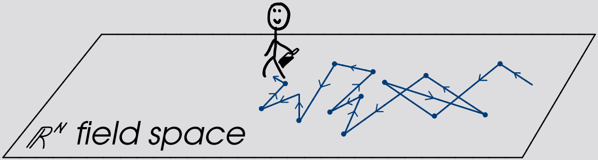




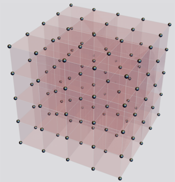
Recap: Monte-Carlo method

"importance sampling" (Monte-Carlo method):
pick out the important (small action) configurations

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int d\phi_1 \ldots d\phi_N \; e^{-S[\phi]} \, \mathcal{O}[\phi] \approx \frac{1}{\mathcal{N}} \sum_{a=1}^{\mathcal{N}} \mathcal{O}[\phi^{(a)}]$$

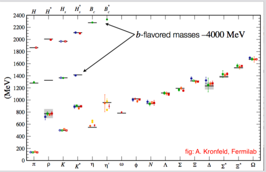


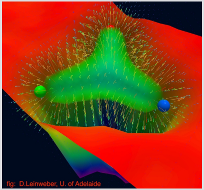
path integral ≈ statistical average with $P[\phi] \propto e^{-S[\phi]}$

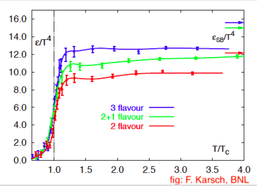


lattice

importance sampling
(Monte-Carlo)





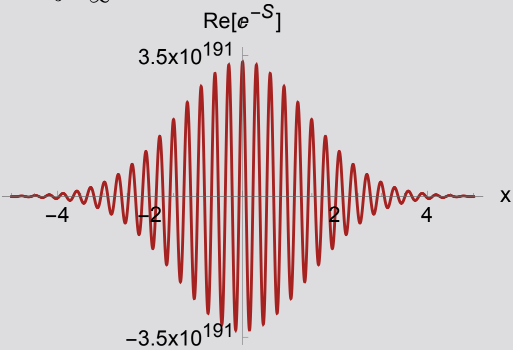


In many cases S, the (effective) action, is complex

- dynamical problems: *out-of-equilibrium, transport, equilibration. . .*
- nonzero density: *many theories with finite chemical potential,QCD phase diagram, critical point, neutron stars, strongly correlated electronic systems, Hubbard model. . .*
- gauge theories with non-zero θ
- matrix models

when the action is complex...

$$\int_{-\infty}^{\infty} e^{-\frac{1}{4}(x+42i)^2}\ dx = 2\sqrt{\pi}$$



"the sign problem"

The sign problem

importance $\propto e^{-S_R}$: *"reweighting"*

$$\langle \mathcal{O} \rangle = \frac{\int d\phi\, \mathcal{O}\, e^{-iS_I}\, e^{-S_R}}{\int d\phi\, e^{-S_R}} \frac{\int d\phi\, e^{-S_R}}{\int d\phi\, e^{-iS_I}\, e^{-S_R}} = \frac{\langle \mathcal{O}\, e^{-iS_I} \rangle_{S_R}}{\langle e^{-iS_I} \rangle_{S_R}}$$

- $\langle e^{-iS_I} \rangle_{S_R} \propto e^{-volume/T}$
- need exponentially large resources

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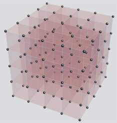
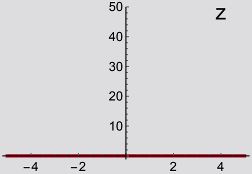
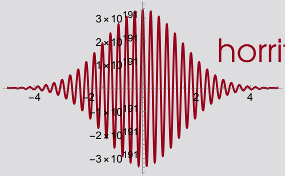


The sign problem

Solving the sign problem

$$\int_{-\infty}^{\infty} e^{-\frac{1}{4}(z+42i)^2}\ dz = 2\sqrt{\pi}$$





horrific sign problem

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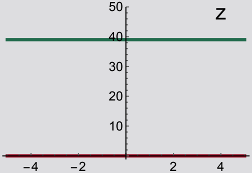
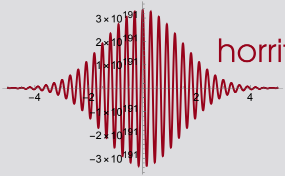


mild sign problem



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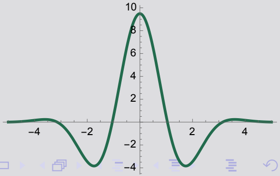

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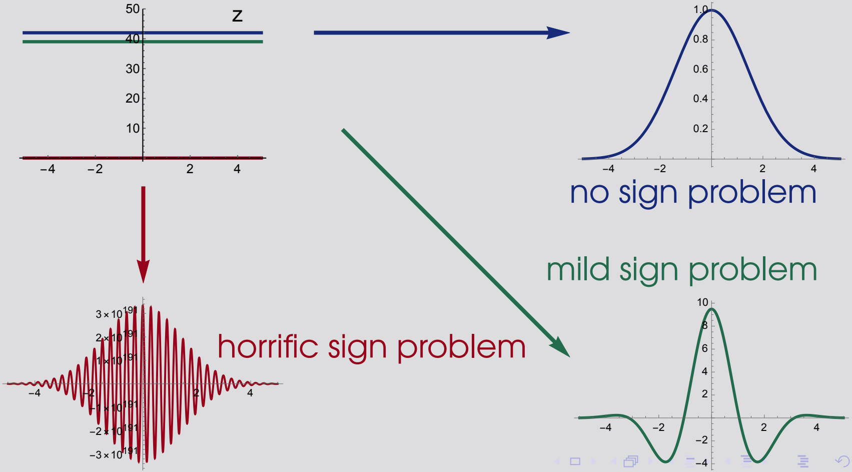
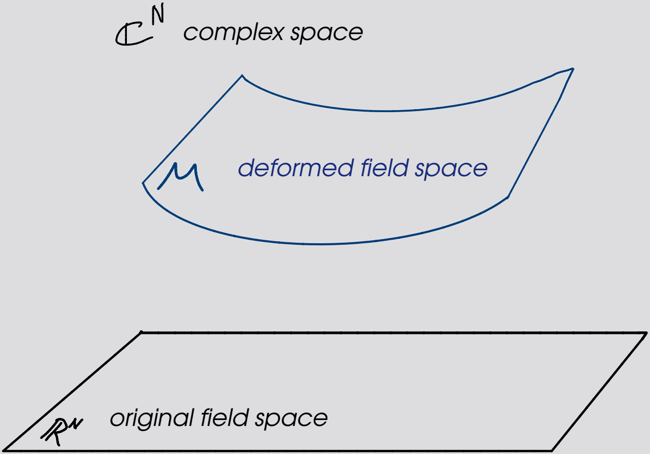


The idea: deform the QFT path integral domain into the complex space and find a better domain where the sign problem is milder

(see also: Makri, Miller; Cristoforetti et al.; Fujii, et al; Tanizaki et al.)

(compare and contrast with complex Langevin: Aarts)

Field theory; good deformation?

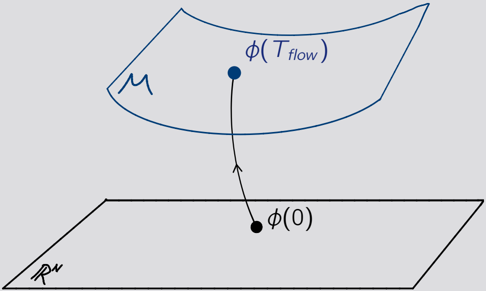


- path integral on $\mathcal{M}$ = path integral on $\mathbb{R}^N$
- sign problem $\mathcal{M} << $ sign problem on $\mathbb{R}^N$

Field theory; good deformation: flow

follow an "equation of motion": flow (Fedoryuk, Witten...)

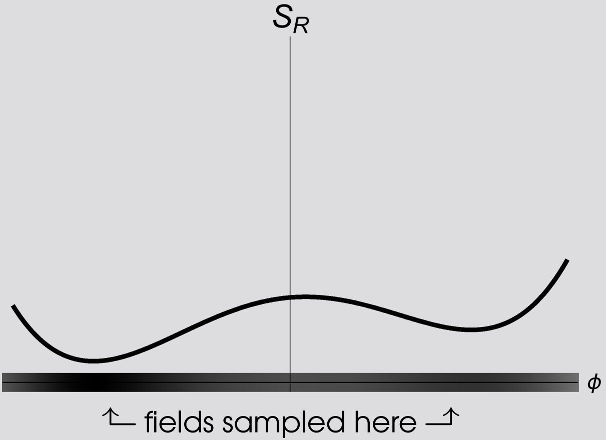
$$\frac{d\phi_a}{d\tau} = \overline{\frac{\partial S}{\partial \phi_a}} \qquad \phi_a = x_a + i y_a \begin{cases} \frac{dx_a}{d\tau} = \frac{\partial S_R}{\partial x_a} = \frac{\partial S_I}{\partial y_a} \\ \frac{dy_a}{d\tau} = \frac{\partial S_R}{\partial y_a} = -\frac{\partial S_I}{\partial x_a} \end{cases}$$



- *steepest ascent for S_R:*
  S_R increases with flow
  (S_R: Morse function)
- *Hamiltonian flow for S_I:*
  S_I remains unchanged

Field theory: good deformations

flow $\to$ S_R increases, S_I remains the same



Field theory: good deformations

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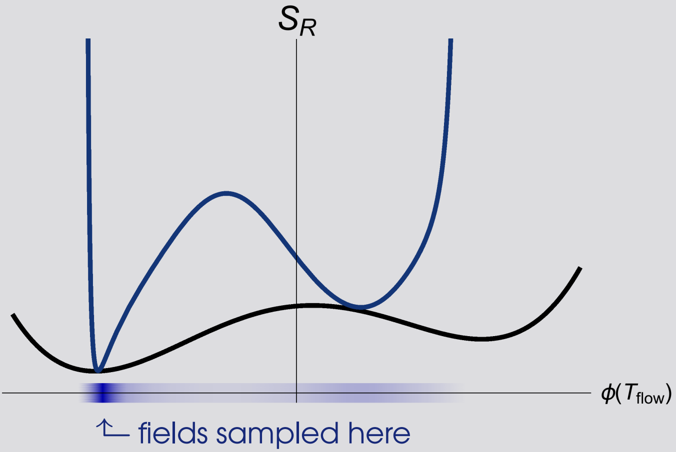
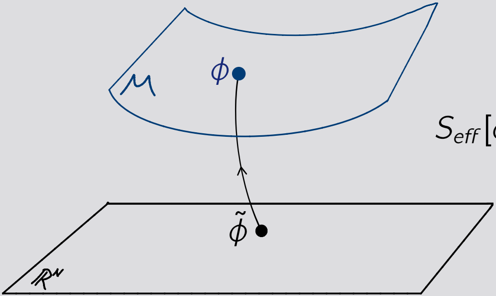


Importance sampling on $\mathcal{M}$

(GB et. al; PRD 93, 014504, JHEP 05 (2016) 053, PRD 93, 094514)

$$\langle \mathcal{O} \rangle = \frac{\int_{\mathcal{M}} d\phi \, \mathcal{O} e^{-S(\phi)}}{\int_{\mathcal{M}} d\phi \, e^{-S(\phi)}} = \frac{\int_{\mathbb{R}^N} d\tilde{\phi} \, \det\left(\frac{\partial \phi}{\partial \tilde{\phi}}\right) \mathcal{O} \, e^{-S[\phi(\tilde{\phi})]}}{\int_{\mathbb{R}^N} d\tilde{\phi} \, \det\left(\frac{\partial \phi}{\partial \tilde{\phi}}\right) \, e^{-S[\phi(\tilde{\phi})]}}$$



$$S_{eff}[\tilde{\phi}] = \mathbb{R}\mathrm{e}\left(S[\phi(\tilde{\phi})] - \log\det J\right)$$

$$\frac{dJ_{ij}}{d\tau} = \overline{\frac{\partial^2 S}{\partial \phi_i \partial \phi_k} J_{kj}}$$

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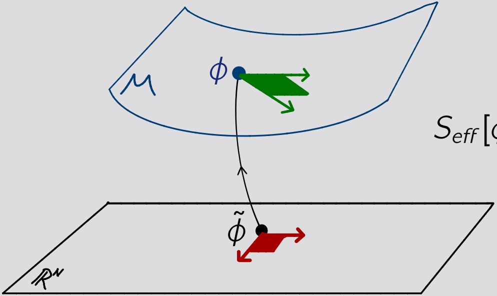

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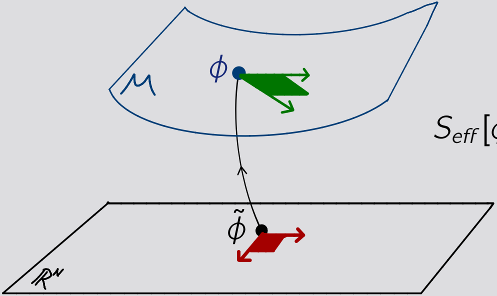
$$\langle \mathcal{O} \rangle = \frac{\int_{\mathbb{R}^N} d\tilde{\phi}\, \mathcal{O} e^{-i\,\mathbb{Im}(S - \log\det J)}\, e^{-S_{eff}}}{\int_{\mathbb{R}^N} d\tilde{\phi}\, e^{-i\,\mathbb{Im}(S - \log\det J)}\, e^{-S_{eff}}}$$



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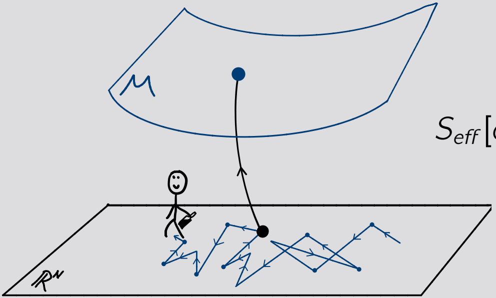

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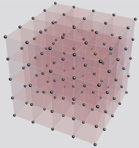
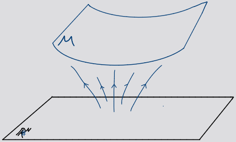
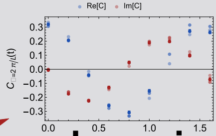
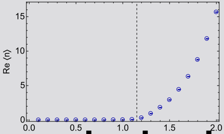


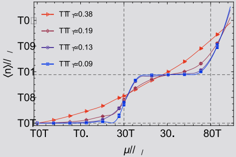




dynamics ✓



many body physics ✓



Our Results

real time dynamics

- anharmonic oscillator
  (Phys. Rev. Lett. 117, 081602)
- 1+1d interacting Bose gas
  (Phys. Rev. D. 95, 114501)

many body systems

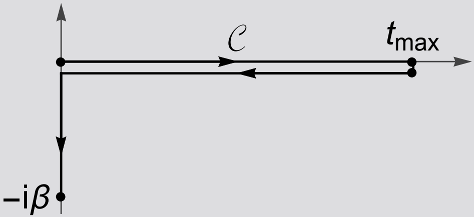
- 1d Hubbard model
  (PRD 93 014504, JHEP 1605 053)
- 2d Thirring model
  (Phys. Rev. D95, 014502)
- 4d interacting Bose gas
  (Phys. Rev. D94, 045017)

Real time dynamics

main object:

$$\langle \mathcal{O}_1(t)\mathcal{O}_2(0)\rangle \quad = \quad Tr[\mathcal{O}_1(t)\mathcal{O}_2(0)\, e^{-\beta H}]$$

viscosity, conductivity, out-of-equilibrium physics...



path integral form:

(Schwinger, Keldysh)

$$S_{SK}[\phi] = \int_{\mathcal{C}} dt\, L[\phi]$$

$$\langle \mathcal{O}_1(t)\mathcal{O}_2(t')\rangle = \frac{1}{Z}\int \mathcal{D}\phi\, e^{iS_{SK}[\phi]}\mathcal{O}_1(t)\mathcal{O}_2(t')$$

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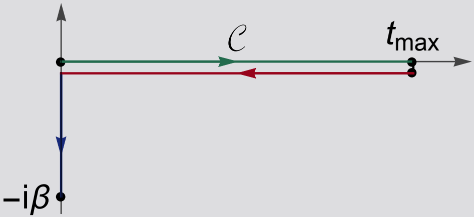

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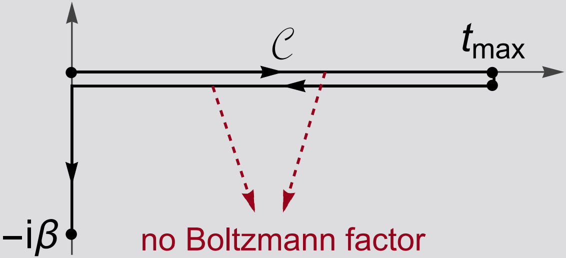
Real time dynamics

the ultimate sign problem!

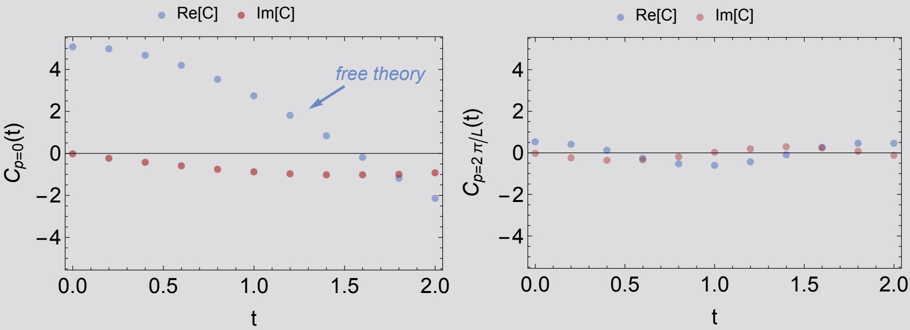


$$\langle e^{i\mathbb{Im}S(\phi)}\rangle_{S_R} = 0$$

Dynamics: 1+1d quantum field theory

interacting Bose gas: $\mathcal{L} = (\partial_\mu \phi)^2 + m^2\phi^2 + \frac{\lambda}{4!}\phi^4$

weak coupling: $\lambda = 0.1$



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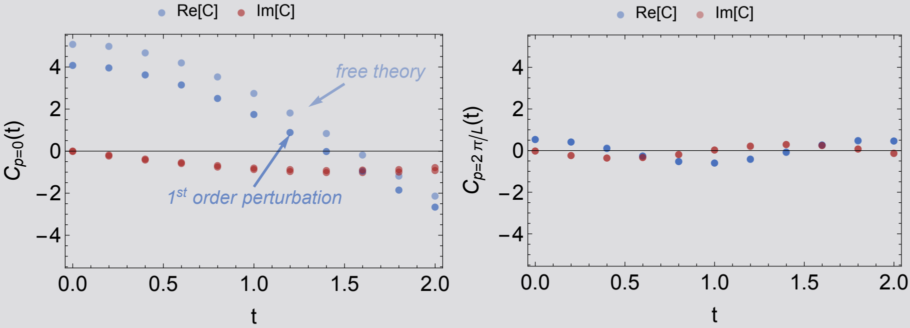

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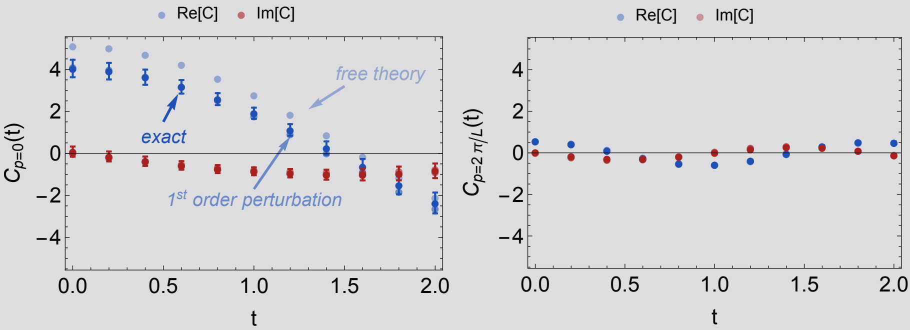

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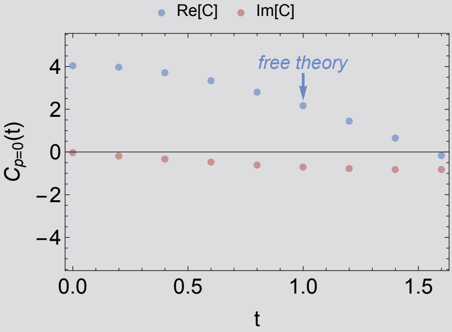

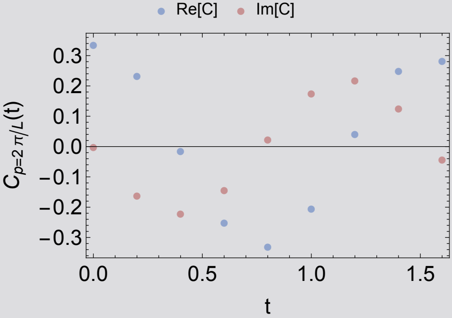

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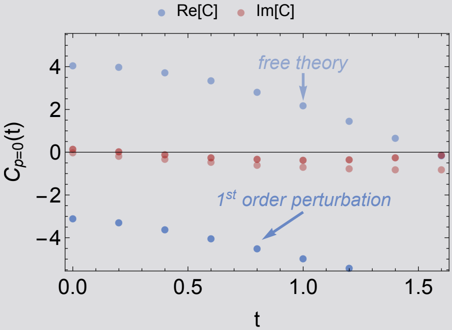

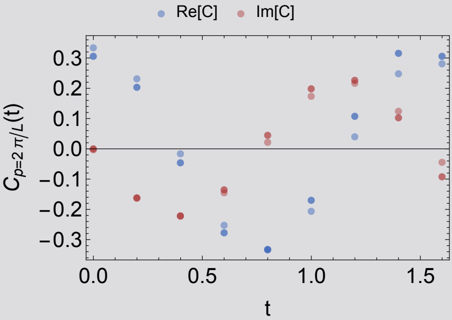

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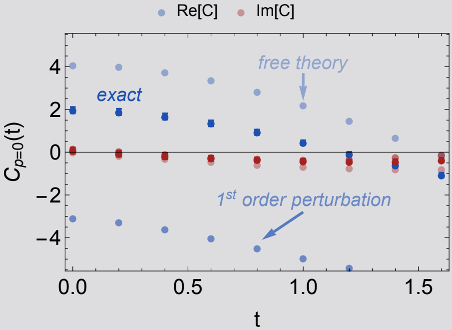

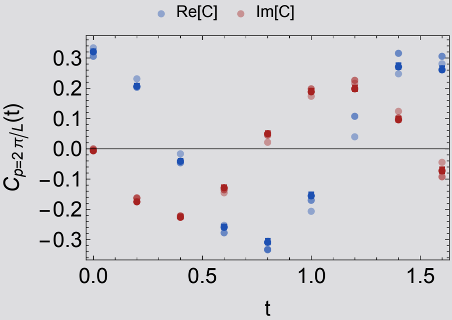

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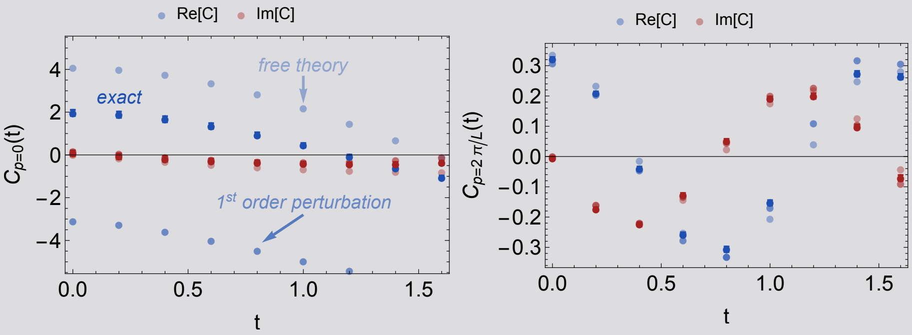
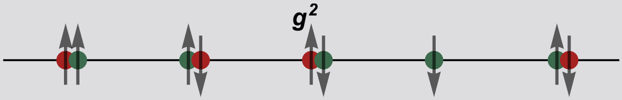


Many body physics: 2d Thirring model



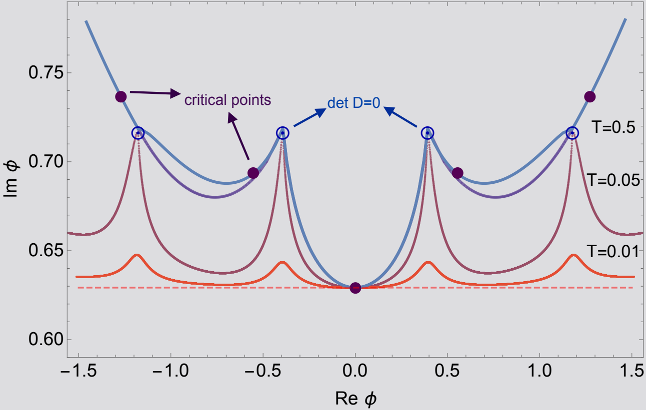
chain of interacting fermions *(quarks, electrons...)*

$$
\begin{aligned}
S &= \int d^2x \bar{\psi}^a \left(\gamma^\mu \partial_\mu + m + \mu\gamma^0\right) \psi^a + \frac{g^2}{2N_f}(\bar{\psi}^a \gamma^\mu \psi^a)(\bar{\psi}^b \gamma_\mu \psi^b) \\
&\to \frac{N_F}{2g^2} \int d^2x\, A_\mu A_\mu + \ln\det(\partial\!\!\!/ + A\!\!\!/ + \mu\gamma_0 + m)
\end{aligned}
$$

- prototype for QCD: *asymptotically free, sign problem at nonzero density*
- a 2d cousin of the Hubbard model

Many body physics: 2d Thirring model

integration manifolds:



projection: $\phi = \frac{1}{L^2} \sum_x A_0(x)$

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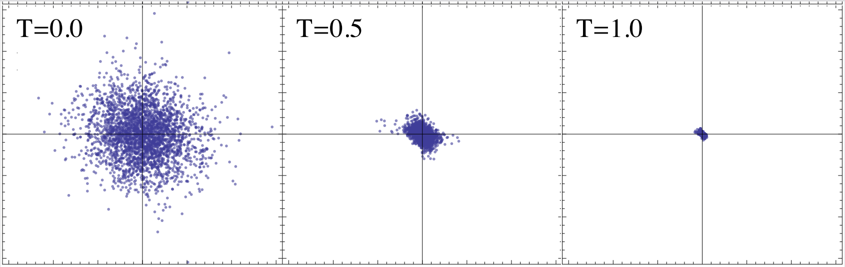
sampled fields ($\tilde{\phi}$):



$\Rightarrow$ more flow $\Rightarrow$

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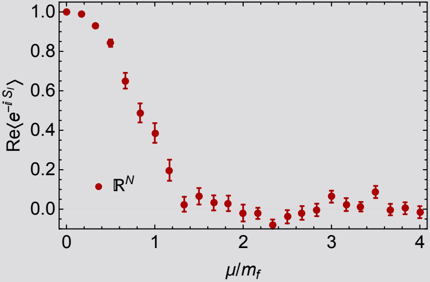
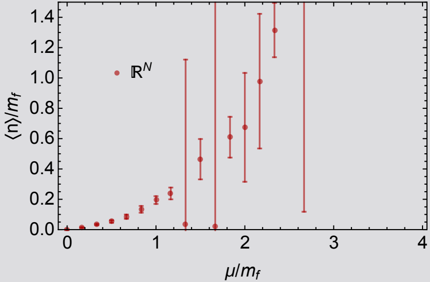
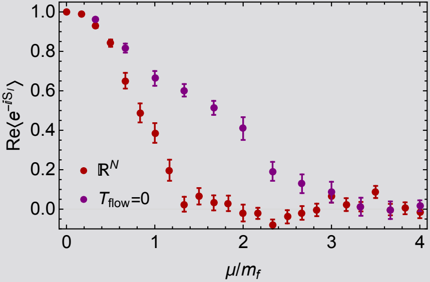


sign problem

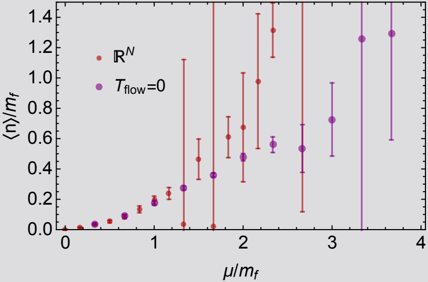


equation of state

Many body physics: 2d Thirring model

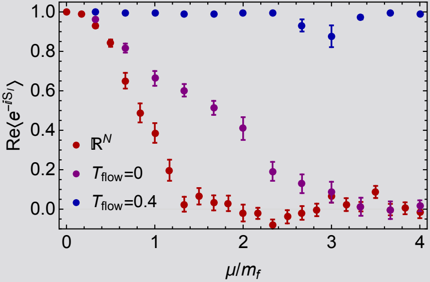


sign problem

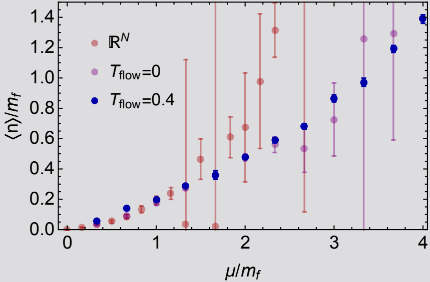


equation of state

Many body physics: 2d Thirring model



sign problem

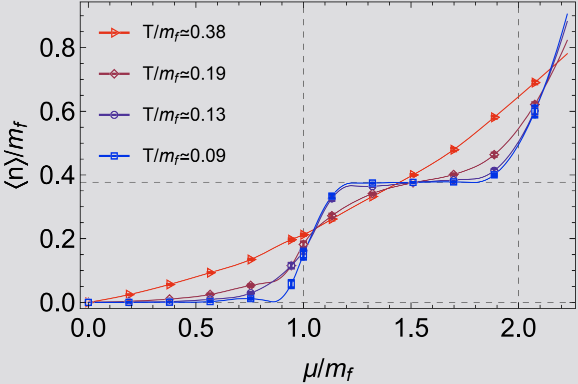


equation of state

Many body physics: 2d Thirring model

equation of state: low temperature limit

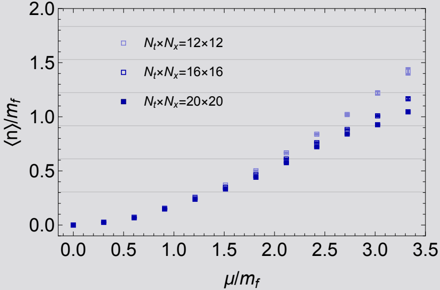
particularly bad sign problem: $\langle e^{-iS_I} \rangle_{S_R} \propto e^{-volume/T}$



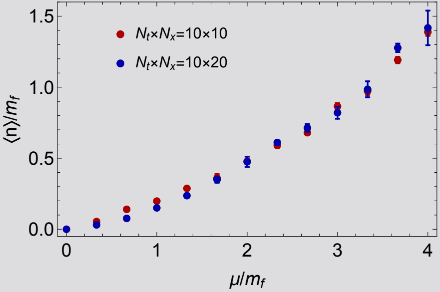
Many body physics: 2d Thirring model

equation of state

continuum limit

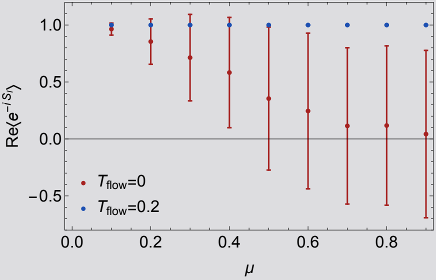


thermodynamic limit

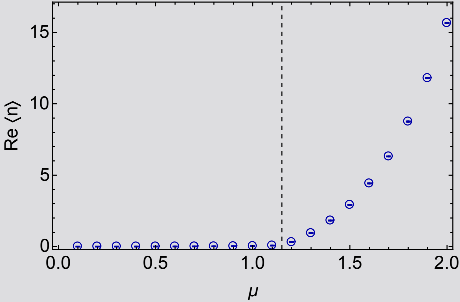


Many body physics: 4d relativistic Bose gas

$$\mathcal{L} = |\partial_\mu \phi|^2 + (m^2 - \mu^2)|\phi|^2 + \mu\left(\phi^* \partial_0 \phi - \phi \partial_0 \phi^*\right) + \lambda|\phi|^4 + h(\phi^1 + \phi^2)$$

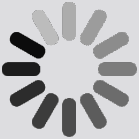


sign problem



equation of state
at low temperature

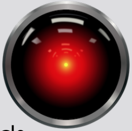
Many body physics: work in progress

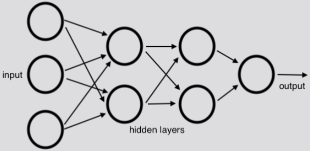


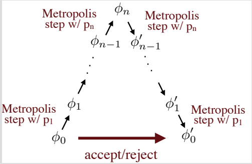
- 3d Thirring model (a closer cousin of Hubbard model)
- 2d QED (a prototype gauge theory)

Future directions

- machine learning:
  - boltzmann machines, faster convergence
  - new manifolds via neural nets, kernel methods...







- tempered transitions
- estimators, pseudo-fermions
- hybrid Monte Carlo

Conclusions

- complexified path integral $\leftrightarrow$ topology
- freedom in deforming the path integration domain makes it possible to ameliorate the sign problem: *many body physics, out of equilibrium, dynamics,...*
- QFT (ferminoic, bosonic), real time ✓
- lots of new directions for future work

“The shortest path between two truths in the real domain passes through the complex domain."

–Jacques Hadamard

The end