

## Tangled in entanglement

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(Received 19 March 2012; accepted 3 August 2012; published online 12 October 2012)

**Abstract:** Conceptual conundrums of quantum mechanics known as instantaneous action at a distance and inseparable wave-particle character are examined by the principle of least action as it was originally given by Maupertuis. When a measurement is understood as a transfer of energy, it follows that the spin of a particle just as the polarization of a photon will remain indeterminate for the observer until the observer receives at least one quantum of action from the object. Thus, it is the dissipative detection that places the photon polarization in the observer's frame of reference. This frame will instantaneously disclose also the polarization of the other photon that emerged from the same radiative decay, provided that the correlation between the two photons has not been perturbed ever since. The wave-particle duality demonstrated by the double-slit experiment can also be understood when a single photon or a single electron is recognized as a flow of energy from a source to a detector. This flow will invariably perturb surrounding energy density, at least the vacuum density. Hence, the total energy density in motion consists of both the particle and the surrounding perturbations. When it channels via two or more paths, the flows through the slits will depend on each other, which will manifest as the inseparable wave-particle character. © 2012 *Physics Essays Publication*. [DOI: 10.4006/0836-1398-25.4.495]

**Résumé:** Le principe de moindre action, comme il était initialement donné par Maupertuis, est utilisé pour examiner les énigmes conceptuels de la mécanique quantique connus sous le nom d'action instantanée-à-la-distance et caractère inséparable onde-particule. Quand une mesure est considérée comme un transfert d'énergie, il s'ensuit que le spin d'une particule de même que la polarisation d'un photon restera indéterminé pour l'observateur jusqu'à quand l'observateur reçoit un quantum de flux d'action de l'objet. Ainsi, il est la détection dissipative qui place la polarisation d'un photon dans le cadre de référence de l'observateur. Ce cadre va aussi révéler instantanément la polarisation du photon qui sorti de la même désintégration radiative, à condition que la corrélation entre les deux photons n'a pas été perturbé depuis. La dualité onde-particule démontré par l'expérience des deux fentes peut également être comprise lorsqu'un seul photon ou un seul électron est reconnu comme un flux d'énergie entre la source et le détecteur. Ce flux va toujours a perturber la densité d'énergie environnante, au moins la densité du vide. Par conséquence, la densité d'énergie totale en mouvement comprend la particule et les perturbations environnantes. Quand il canalise par le biais de deux ou plus chemins, les flux à travers les fentes dépendent l'un a l'autre, qui se manifestera comme caractère inséparable onde-particule.

Key words: Action at a Distance; Double-Slit Experiment; Evolution; Quantum Paradoxes; Principle of Least Action.

### I. INTRODUCTION

Spooky action at a distance is one of the weirdest quandaries of quantum mechanics. The phenomenon has been chewed up so thoroughly that it perhaps leaves a bad taste even to attempt to take a bite of it. Nevertheless, to have a fresh look at the correlation without a force carrier, let us imagine how the perplexing phenomenon<sup>1-3</sup> would appear to a rookie knowing only basic physics. He would specifically like to understand why the measure-

ment of a photon's polarization will instantaneously reveal also the polarization of its mate photon that has emerged from the same radiative decay.

At first it may appear unproductive to abandon the established formalism of quantum mechanics, but that theory is limited to describe conserved systems, just as statistical mechanics is limited to account for closed or stationary systems.<sup>4,5</sup> When the quantum theory, just as the classical statistical mechanics, forces probabilities to sum up to unity, it will fail to account for the change in energy due to the detection where at least one quantum is either acquired from an object system to an observer or

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FIG. 1. (Color online) A decay process (at the center) outputs two photons (blue/left and red/right), which propagate in the opposite directions. As long as neither one of the two is subject to unequal forces, the polarization of one photon will remain intact relative to its mate but indeterminate relative to any external frame of reference (depicted as an axis system/blue). The detection will place the polarization of one photon in the reference frame relative to the observer, and the same frame will also disclose instantaneously the other photon provided that the correlated pair has remained unperturbed ever since it emerged from the decay.

vice versa. In contrast the principle of least action in its original form à la Maupertuis<sup>6</sup> is able to account for the flows of quanta from the system to its surroundings (or vice versa), while complying with the conservation of quanta. This universal principle to consume free energy in least time is familiar to a rookie because it describes flows of energy that will level off energy differences between the system and its surroundings in the same way as heat will flow from hot to cold. Maupertuis's equation for evolving systems is distinct from Lagrange's equation of the variational principle that applies to conserved, i.e., stationary systems. Admittedly, Maupertuis's version of the least-action principle is today widely considered to be ill formed<sup>7</sup> because it is noncomputable.<sup>8</sup> When the flows of energy consume their driving forces, i.e., energy differences, variables cannot be separated to solve the differential equation of motion. For example, when a rock rolls down from a hill top to a valley bottom, the height difference that drives this natural process will diminish due to the motion itself. When the change in the energy landscape is negligible, a prediction is precise enough, but in general natural processes are noncomputable. Thus, the Maupertuis principle, despite its equation of motion being noncomputable, is an accurate description of the natural processes in which the system evolves from one state to another due to an influx of energy from its surroundings or an efflux to its surroundings.

The simple and ubiquitous imperative of the least-time free-energy consumption provides a fresh look into a seemingly complicated problem of the correlation without a force carrier. The natural principle has been recently derived as an equation of motion from statistical physics of open systems and applied in analyses of diverse problems.<sup>9–14</sup> At first sight the following physical portrayal may appear to some experts too concrete and simple because customarily much of quantum mechanics is communicated using abstract concepts and illustrated by gedanken experiments. Here, we deliberately limit our description to plain observations that can be communicated and analyzed using basic concepts. Conversely, we will avoid thought experiments and concepts whose correspondence to reality is ambiguous to us. By this strictly naturalistic description, we hope also to avoid

entering obscure philosophical discussions related to the conundrums of quantum mechanics over the years.

## II. PHASES EXIST RELATIVE TO REFERENCES

Detection, when described as a natural process, will require some flow of quanta between an observer and an object.<sup>9,15</sup> In view of that, the observer must capture at least one quantum of action, i.e., one photon, from the object. Alternatively the observer may perceive the object by contributing at least one photon to it. Thus, our rookie reasons that the measurement is an energy transduction process that will move the object from its initial state either to a final state that is down in energy or to another final state that is up in energy relative to the initial state—which one of the available alternatives is valid depends on whether the observer is in the absorptive or emissive state relative to the object. For example, when the observing detector is at a ground state, it may accept a photon from the object provided that the object is in an excited state to donate the photon. Conversely, when the observing detector is in an excited state, it may emit a photon to the object provided that the object is in the ground state to absorb the photon. When both the observer and the object are in the ground state or both are in the excited state, there cannot be any flow of quanta; hence, nothing can be detected.

Notably, because the rookie regards the measurement as an energy transduction process, he finds no need for the prevailing presumption that the unobserved, i.e., unperturbed object would be in some superposition of states. Therefore, he deduces that the indeterminacy in the outcome of a measurement is not contained in the initial state of the object as such but follows from randomness in the phase between the object and observer. When the phase between the object's and the observer's frames is arbitrary, the flow of energy will direct randomly along a path among the alternative absorptive and emissive transitions. Thus, the observation, when uncorrelated with the object's motions, will bring the object's motions at random from the initial state to a final state. Thus, a series of uncorrelated measurements will deliver a distribution of outcomes. Conversely, when the phase between the object's and observer's frames is not arbitrary, correlated phenomena will appear.

When the measurement is understood as an energy transduction process, it follows that the polarization of a photon will be established relative to the frame of observation (see Fig. 1). In other words, the spin of a force carrier remains indeterminate, as stated by the Einstein–Podolsky–Rosen (EPR) paradox, until the detection relates it to the observer's frame. Likewise for a correlated pair of two photons, the indeterminacy of polarization with respect to the observer will prevail, but the mutual polarization of one photon relative to the other photon will survive as long as the observation or any other process will perturb the frame of the two photons that resulted from the same decay. In the absence of forces, the initial phase ( $\varphi = \pi$ ) between the two photons will remain

invariant, i.e., at a constant value in accordance with Newton's law of constant motion ( $d_t\phi = 0$ ).

The entanglement, i.e., the correlation between the two particles is fragile. When the two particles experience unequal gradients in energy, the degree of order will invariably decrease because the ensuing flows of energy differ for the two particles in propagation. In view of that, to preserve the crucial correlation, i.e., to retain the relative phase between the pair of particles does not require the presence of a force carrier whose coupling, in fact, would destroy the correlation. When one of the two correlated photons having opposite polarizations is related to the observer's frame in the detection, the same frame, irrespective of the outcome, will apply instantaneously to the other photon having the opposite polarization—provided that the mate has not been perturbed in the meantime by any other process.

The idea in knowing instantaneously one from the other is comparable when drawing one marble out of two having opposite colors from a sealed bag for inspection of its color. Irrespective of the outcome, the color of the other marble having the opposite color is known instantaneously—provided that the mate has not been changed in the meantime by anyone. To deduce the color of one from the other does not depend on particular colors, only that they are the opposite for the two marbles. Likewise, to deduce the polarization of one photon from the other does not depend on particular polarizations, only that they are opposite for the two photons, which they have to be in order to conserve the angular momentum in the decay that produced the two quanta.

Thus, the rookie concludes that indeed no flow of energy, i.e., a causal connection, is required to deduce instantaneously the spin of the other particle from the one referenced at the measured site provided that their mutual orientation has remained the same ever since the two emerged as a correlated pair.<sup>16</sup> However, to reveal that the detected photons, in fact, were a correlated pair, communication from one site of a recording to the other site of detection is needed. The mandatory message can at best flow at the speed of light because any information must have some form of physical representation.<sup>17,18</sup> Hence, the rookie reasons that a flow of information is the flow of energy. The received quanta will change the receiver state from that where recordings of a photon polarization are random to that where correlations are recognized on the basis of the information given about the mate photon. Likewise, it does not mean much to pick a series of marbles of red, blue, and white from a bag until one is told that another series of green, yellow, and black marbles, the sequence of opposite color, were also drawn from the same bag.

The resolution of the EPR paradox as given above does not involve hidden variables, merely concepts of energy and time that together define an action  $S = \int \mathbf{p} \cdot d\mathbf{x}$ , which integrates momenta  $\mathbf{p}$  along paths  $d\mathbf{x} = \mathbf{v}dt$ , where the kinetic energy  $2K = p^2$  landscape is in motion with velocity  $\mathbf{v}$  during time  $dt$ .<sup>6,9,10</sup> An observation, just any

other flow of energy, will drive the energy landscape in evolution from one stationary state toward another<sup>9,10</sup> by consuming at least the quantum of action, corresponding to the Planck's constant  $h$  (Fig. 1). The indeterminacy  $\Delta 2K\Delta t \geq h$  is inherent in the detection because no state can be determined without causing a change of the state by at least  $h$ . A macroscopic system is not perturbed much from its initial state when losing few quanta, but a microscopic system will suffer severely, eventually going extinct when losing the very last quantum of action to the observing surroundings. This impact of detection on the object has been phrased memorably by Pascual Jordan, "Observations not only disturb what is to be measured, they produce it . . ."<sup>19</sup>

### III. INSEPARABLE FORCES AND FLOWS

Let us dissect another fundamental subject of quantum quandaries, the double-slit experiment, from the fresh viewpoint provided by the old principle of least action. Our rookie would specifically like to understand, why the wave and particle character of light and other forms of energy cannot be separated from each other.

According to the principle of least action, flows of energy will act to diminish energy density differences in least time. Thus, it follows that a flow of energy, such as a stream of photons or electrons from a high-density source, will disperse along the least-action paths, for example, those passing via two slits. The flows of energy over a time interval  $t$  will consume the driving forces due to the scalar  $U$  and vector  $\mathbf{Q}$  potential differences, and the balance is maintained by a change in the kinetic energy  $2K$ . The conservation among the three forms of actions  $2Kt = -Ut + \mathbf{Q}t$  was conjectured a long time ago.<sup>20,21</sup> The balance equation is easy to recall in a differential form, where an electron is accelerating down along an electric field and emitting light, i.e., dissipating energy to the surroundings down along the vector potential gradient orthogonal to the electron's directed path.<sup>9,22</sup> Curiously though, when energy disperses from a source down along two or more paths, the natural process will be intractable.<sup>8</sup> In other words, the path-dependent process cannot be integrated to a closed form because the end point of trajectory cannot be known beforehand. The derivatives at the branching points are inexact, i.e., the tangent is ill defined at a fork of path because the quantum is indivisible. The flows that consume the same source of energy are interdependent because when a flow by the mere act of flowing is decreasing the common driving density difference, i.e., the gradient that also fuels other flows, these flows will be affected as well and vice versa.<sup>14</sup> In general, natural systems when changing their states from one action to another are non-Hamiltonian systems. Only when a single path is provided, can forces and flows be separated from each other to allow integration to a closed form. Also the stationary-state trajectories are deterministic because energy of the system, i.e., the Hamiltonian is invariant.



FIG. 2. (Color online) Photons (or electrons) are energy densities (depicted as waves with contrast variations) that propagate via two available paths (openings in the red wall) down along the energy density gradient between a source (up) and a sink (down). A propagating energy density will perturb the surrounding vacuum energy density and thereby also induce energy density differences that, in turn, act as driving forces. When the density perturbation extends from one slit over to the other, the two flows that are consuming it will depend on each other. Because the flow of energy density through one slit will affect the density difference across the other slit, the paths are interdependent and give rise to inseparable particle-wave character (<http://www.falstad.com/ripple>).

The progress of a natural process, where quantized flows of energy are leveling off the density differences, is measured by a change  $d_t P = d_t \int \psi^* \psi dx$  in the probability  $P$ . So the probability is physical. It is a measure of the system's status in energetic terms. The wave function  $\psi(\mathbf{x}, t)$  is often regarded merely as abstraction, but here it turns out to be a particularly fitting physical formalism, via its mutually orthogonal spatial and temporal variables, to describe a flow of energy density from one locus of energy density along  $\mathbf{x}$  to another locus during time  $t$ . When the flows level off the density differences, the wave functions will change. Eventually the natural process will attain the state where all forces, including those imposed by the surroundings, are perfectly balanced. In other words, at the free-energy minimum state, the energy landscape has no net curvature.<sup>9</sup> Then, the system has arrived at a thermodynamic steady state  $d_t P = 0$ , where the opposite circulations of energy densities  $\psi$  and  $\psi^*$ , as familiar from Kirchoff's law, are equally abundant on their common trajectories. The conserved quantities of the stationary state, most notably mass, relate to the symmetry group of the action's path via Noether's theorem.<sup>23</sup> Because the energy of the stationary state is conserved, there is a norm, and hence a unitary transformation can be found that will remove the time dependence altogether. Thus, dynamics of a Hamiltonian system is completely reversible, whereas the consumption of free energy results in an irreversible process.<sup>24</sup>

It does not occur to the rookie that some form of an energy density could possibly be a pointlike singularity that would fit into its surrounding energy densities without causing any perturbation. On the contrary, a photon has its wavelength, and the electron's finite magnetic moment and charge also imply some finite-sized circulation whose energy density will invariably perturb surrounding energy densities.<sup>10</sup> This, of course, manifests itself, e.g., as an electric field. Because no space is without some energy density,<sup>11,12</sup> at least the vacuum density will be perturbed

by the flows of densities, for example, in the form of light or electrons that propagate down along the available paths from a source to the sink that acts as a detector. The nonzero values of permittivity and permeability that define the speed of light  $\epsilon_0 \mu_0 = c^{-2}$  and impedance  $\epsilon_0 / \mu_0 = Z^{-2}$  reveal that the all-around present vacuum density is not zero. Indeed the photon-embodied physical vacuum was proposed already early on, and it has maintained interest ever since.<sup>25-28</sup> The finite universal energy density manifests itself in perihelion precession,<sup>11</sup> galactic rotation,<sup>12</sup> propagation of light,<sup>13</sup> Casimir effect,<sup>29</sup> and Aharonov-Bohm experiment.<sup>30</sup> In particular, the dynamic Casimir effect<sup>31</sup> reveals that the vacuum density comprises photons, but when their phases are on the average opposite to each other, i.e., random, there is no electromagnetic field. When an electron propagates through an energy density  $U$  that is applied on the top of the vacuum density, the interference pattern will acquire an additional shift  $\Delta\phi = Ut/\hbar$  in the phase, known as the Berry phase.<sup>9,32</sup> Conversely, it follows that if the vacuum density were zero, there would be no interference pattern without applied potential. Thus, the rookie understands that as long as the two slits are within the spread of an energy-density perturbation, the flow of energy density through one slit will depend on the flow through the other slit because both streams consume the common density difference. When one slit is conducting, the density difference across the other slit will also change and affect the other flow and vice versa (see Fig. 2). Therefore, inseparable interference effects will arise, when the coherent flows recombine after they have taken two or more paths to maximize the overall dispersal of energy in the least time.<sup>33</sup> Moreover, any attempt to sniff how the flows of energy density distribute between to the two slits, will require, just as any measurement, some flow of energy between the object and its observer. This coupling, in turn, will obscure or even destroy the interference pattern, just as any correlation, when contributing incoherently to the energy dispersal process.

The interdependence between the flows of energy densities and the energy-density differences as their driving forces is familiar also from the three-body problem.<sup>34</sup> When there are three or more degrees of freedom, irreversible processes are intractable because the forces and flows cannot be separated from each other for the integration to a closed form.<sup>35</sup> Only when a single path is provided, i.e., when there are only two degrees of freedom, will the energy dispersal process be deterministic. Physics mostly focuses upon these special cases of bound actions, known as the Hamiltonian systems, where the forces can be separated from the flows, and the motions can be tracked by integration.

#### IV. ACTIONS CONSTITUTE SPACE AND TIME

Undoubtedly, there are many more challenging questions to inspect, for example, quantum Zeno effects<sup>36,37</sup> where flows of energy in detection correlate with the changes of state of the object's with (periodic) energy density variations of the observer by the universal law of

least-time free-energy consumption. In these cases, too, reality can be seen in a holistic way so that diverse systems take part in the overall process of energy dispersal by interacting with their surrounding systems.<sup>38</sup> No system is without some surroundings, at least that of the universal surroundings known as the vacuum. Therefore, everything depends on everything else, in concert with Newton's laws. It is the energy differences between the system and its surroundings that will drive changes of state of both the system and its surrounding system. When the system evolves from one state to another, the boundary conditions defined by the surroundings will change too because it is the surroundings that will either supply or draw the quanta needed for the system's change in energy.

Physics, like any other discipline, is beguiled by ambiguity in its most central concepts, most notably space and time. Commonly these are used as axioms, but, of course, they, too, do deserve physical presentations to be solid cornerstones of our description of reality. According to the principle of least action, any element in space is a stationary action composed of one or multiples  $\hbar$ .<sup>10</sup> This spatially localized energy density is surrounded by other actions. Their mutual energy density differences are the driving forces that will diminish with time whose flow, in turn, is recognized as a flow of energy density. An element of time is an open action composed of one or multiples  $h$ . The open actions carry energy from high-density closed confinements to others of lower densities. These natural processes, even when the equation of motion is intractable, will naturally select from the available variation the least-time paths of dispersal, known also as geodesics.<sup>12,39,40</sup> Our rookie, when facing the diversity of natural phenomena, is easily lost in case-by-case reasoning garnished with diverse formulas but prefers to apply the general principle of least action that may be too transparent to catch the eyes of those tangled in entanglement.

## V. CONCLUSIONS

Quantum mechanics, like many other theories of physics, is constructed to be computable by imposing a unitary condition so that energy of the system is conserved. Hence, the theory cannot describe a change of state, e.g., that due to an observation, where the system either absorbs at least one quantum from its surroundings or emits one quantum to its surroundings. Moreover, the stationary-state description provided by the quantum mechanics does not explicitly involve surrounding densities of energy that invariably include at least the vacuum density. Hence, the theory cannot describe understandably interference phenomena, e.g., the double-slit experiment, where both the system and its surroundings are involved. Due to these imposed but artificial constraints embedded in the quantum mechanics, we argue that the perplexing phenomena, known as instantaneous action at a distance and inseparable wave-particle character, are merely conceptual conundrums of quantum mechanics but present no true problem of understanding when examined by the principle of least action given in its original form à la Maupertuis.

The Maupertuis principle, despite its equation of motion being noncomputable, is an accurate description of the natural processes where the system evolves from one state to another due to an influx of energy from its surroundings or an efflux to its surroundings.

## ACKNOWLEDGMENTS

We thank Aki Kallonen, Stanley Salthe, and Juha Samela for insightful corrections.

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