Least-time paths of light

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ABSTRACT

The variational principle in its original form á la Maupertuis is used to delineate paths of light through varying energy densities and to associate shifts in frequency and changes in momentum. The gravitational bending and Doppler shift are in this way found as mere manifestations of least-time energy dispersal. In particular, the general principle of least action due to Maupertuis accounts for the brightness of type 1a supernovae vs. redshift without introducing extraneous parameters or invoking conjectures such as dark energy. Likewise, the least-time principle explains the gravitational lensing without the involvement of additional ingredients such as dark matter. Moreover, time delays along curved geodesics relative to straight paths are obtained from the ratio of the local to global energy density. According to the principle of least action the Universe is expanding uniformly due to the irrevocable least-time consumption of diverse forms of bound energy to the lowest form of energy, i.e., the free electromagnetic radiation.

Keywords: cosmological principle; energy dispersal; evolution; Fermat's principle, geodesic; the principle of least action

1 INTRODUCTION

A ray of light takes the path of least time. This well-known principle by Pierre de Fermat is a special form of the general principle of least action (De Maupertuis 1744; Tuisku, Pernu & Annila 2009). Namely, light, as any other form of energy in motion, will naturally select the path of propagation that will maximize the dispersal of energy (Kaila & Annila 2008). The derivation of Snell's law by the least-time principle is a familiar textbook example (Alonso & Finn 1983). However, does the same variations principle govern also light's passage through the expanding Universe from a high-density distant past to the low-density near-by present? The answer will illuminate interpretation of supernovae data (Goldhaber & Perlmutter 1998; Garnavich et al., 1998) that seems to signal for a faster expansion than is expected on the basis of known forms of energy – possibly due to dark energy (Perlmutter 2003).

Moreover, light caught bending when passing by the Sun, is a famous proof of general relativity (Einstein 1911; Berry 2001). However, does the least-time principle also govern light's refraction when passing by all gravitating bodies? The answer will explain galactic gravitational lensing (Blandford & Narayan 1992) that seems to be stronger than expected on the basis of luminous matter – possibly due to dark matter (Goldsmith 1991).

According to the principle of least action, light will follow the path where the integrand of variations is at a minimum (Feynman 1965). Customarily the integrand is a Lagrangian which, as a conserved quantity, can be used to determine stationary paths of stationary-state systems. However, the expanding Universe is an evolutionary system where light must on its way adapt to changing circumstances. Likewise, light must adjust its energy to varying surroundings, when passing by a local variation in the universal energy density. Enlightening light's least-time paths through changing surroundings is the objective of this study. Therefore, rather than using the conserved Lagrangian form of the action principle (Kovner 1990), its original form á la Pierre Louis Moreau de Maupertuis will be used here. In the general form of the action principle kinetic energy is integrated over time, or equivalently momentum is integrated over the path. This form has for long been shunned but recently it has been derived from the statistical physics of open systems (Kaila & Annila 2008; Sharma & Annila 2007; Annila 2010a). Subsequently it has been used to describe diverse evolutionary processes (Mäkelä & Annila 2010; Annila & Salthe 2009; Annila & Salthe 2010; Annila, 2010b).

2 FREQUENCY SHIFT IN REFRACTION

The starting point for the analysis of light's trajectories through evolving surroundings is the conservation of energy. Continuity requires that a change in kinetic energy d_t2K must balance changes in the scalar ∂_tU and vector ∂_tQ potentials. Specifically, when light of energy $Q = hf$ at frequency f moves from the universal gravitational potential known as the free space to the local gravitational potential $U = -GM_khf/c^2r$ due to a body of mass M_k, the change in kinetic energy $2K = hf_k$ at f_k must balance the change in U and Q

$$d_t 2K = -\partial_t U + \partial_t Q \Leftrightarrow d_t mv^2 = -\partial_t \frac{GmM_k}{r} + \partial_t mc^2$$
$$\Rightarrow \frac{1}{n^2} = \frac{v^2}{c^2} = 1 - \frac{GM_k}{c^2 r} \Rightarrow f_k = nf \qquad (1)$$

where the squared index of refraction $n^2 = c^2/v^2 = (1 - GM_k/c^2r)^{-1} \approx (1 + GM_k/c^2r)$ contains the ratio of the universal $R = c^2/a$ to a local $r = v^2/a_k$ radius of curvature. The universal and local surroundings are defined by the universal $a = GM/R^2$ and local $a_k = GM_k/r^2$ acceleration resulting from the respective universal $M = \Sigma m_i$ and local M_k source of energy density. In general, when light crosses from energy density k to energy density j, the conservation of energy will require that the frequency will shift according to $d_t 2K_k = d_t 2K_j$, i.e., $f_k = (n_k/n_j)f_j$ and that the direction of propagation will change in accordance with the change in momentum $d_t\mathbf{p}$ (Fig. 1).

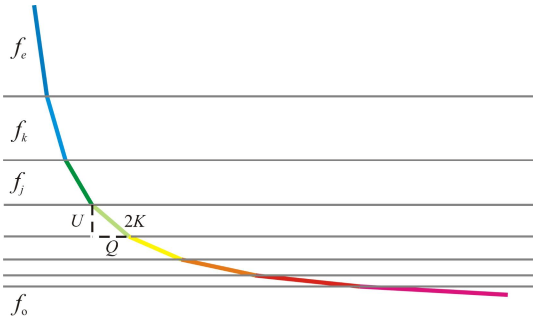

Figure 1. Light will shift from blue to red when traversing from energy-dense environs specified by index n_k to sparser surroundings given by n_j in relation to the universal reference surroundings of index $n^2 = c^2\varepsilon_o\mu_o = 1$. For a certain energy hf_k there is no partition at a boundary because beyond the limit of total internal reflection $\theta_j = \pi/2$ at $n_k\sin\theta_k > n_j$ the energy density in the j-medium falls short to support the propagation. Thus the lowest frequencies deflect least and distribute most uniformly.

When light traverses through a stratified medium, it will be subject to a series of refractions where frequency and direction of propagation will shift progressively (Crawford 1968). Specifically, light that has emerged from the high-density surroundings of the distant past and arrives at low-density surroundings of the present moment has accumulated a total frequency shift, known as the Doppler shift, from the series of refractions. The passage through the diluting universal surroundings that started by an emission $Q = hf_e$ at a rate f_e in that time universal density bounded within radius r_e will finish at the absorptive observation at the rate f_o in today's universal density that is confined within R. The dilution series

$$f_k = \frac{n_k}{n_{k-1}} f_{k-1} = \frac{n_k}{n_{k-1}} \frac{n_{k-1}}{n_{k-2}} f_{k-2} \Rightarrow \frac{f_e}{f_o} = \frac{c}{v_e} = n = 1 + z \qquad (2)$$

yields $z = (f_e - f_o)/f_o = (1 - GM_e/c^2r_e)^{-1/2} - 1$ for the total redshift. In other words, light will adapt on its way from dense to dilute surroundings by depositing its decreasing momentum $\mathbf{p}$ on an increasing wavelength λ.

According to the natural principle of maximal energy dispersal the shift in frequency results from the change in the surrounding energy density, whereas customarily the frequency shift has been seen to signal kinematics, e.g., relative motion between a source and a sink. However, the two views are not different since non-stationary motions, i.e., evolution will invariably cause changes in energy density. When the source is receding, the flux density of received energy at the sink is decreasing. Conversely, when the source is approaching, the flux density at the sink is increasing. The value in describing the frequency shifts as resulting from the changes in energy density is that there is no need to make distinctions between motional, gravitational or cosmological red or blue shifts. Irrespective of the particular process, all frequency shifts and changes in momentum are understood to stem from changes in energy density.

3 APPARENT MAGNITUDE OF STANDARD CANDLES

According to the principle of maximal energy dispersal the frequency shift of light from a distant source results from the rate at which the Universe is expanding. To quantify the rate of this process the type Ia supernovae serve to determine distances beyond our galaxy since their absolute magnitudes are expected to be the same. All explosions ignite at the same mass, hence luminosity $L = d_tQ$ is a standard (Colgate 1979). At the explosion the energy $Q = hf_e$ dissipates at the rate f_e and disperses from the past universal surroundings of radius r_e through an area $4\pi r_e^2$ onto the larger and larger universal surroundings. Ultimately when light has passed through the area that has increased up to $4\pi(R - r_e)^2$ it will be at absorbed at the rate f_o when observed (Fig. 2).

The inverse square law relates the observed flux F_o at the luminosity distance $D_L = R - r_e$ between the current radius $R = cT$ of the Universe and its past radius r_e when the explosion took place according to

$$F_o = \frac{L_o}{4\pi D_L^2} = F_r \frac{L_o/4\pi D_L^2}{L_r/4\pi D_r^2}$$
$$\cong F_r \frac{hf_e f_o r_r^2}{hf_e f_e D_L^2} = F_r \frac{1}{1+z} \frac{r_r^2}{R^2} \frac{(1+z)^2}{z^2} = F_r \frac{r_r^2}{R^2} \frac{1+z}{z^2} \qquad (3)$$

where the definition $z = (\lambda_o - \lambda_e)/\lambda_e$ is used as $z = (R - r_e)/r_e$ so that $D_L = Rz/(1+z)$. Customarily z of the reference flux F_r that is received from a supernova at D_r = 10 parsec is considered negligible, i.e., $f_r = f_e$ and $r_r = D_r$.

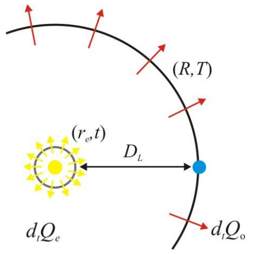

Figure 2. Light disperses from a supernova explosion (yellow) in least time through spherical shells of increasing radius from the time t when the Universe had a radius r_e to the present-time T when the Universe is of a radius R. The flux of emitted energy $d_tQ_e = hf_ef_e$ dilutes on its way from the past, dense universal surroundings to the present, sparse universal surroundings where the flux $d_tQ_o = hf_ef_o$ is absorbed at the site of detection (blue).

Since the supernovae are dispersed far and wide from us, the observed fluxes vary over several orders of magnitude. Therefore the brightness data is customarily reported on a logarithmic scale so that the apparent magnitude $m = M_r - 2.5\log(F_o/F_r)$ is given in terms of the absolute magnitude $M_r = 2.5\log(F_o/F_r)$ of the reference supernova. Then the cosmic distance ladder (Hogg 1999) can be given by the distance modulus

$$\mu = m - M_r = -2.5\log\left(\frac{L_o/4\pi D_L^2}{L_r/4\pi D_r^2}\right) + K = 2.5\log\left(\frac{R}{r_r}\right)^2 + 2.5\log\left(\frac{z^2}{1+z}\right) + K \quad (4)$$

where the K-correction, having approximately the form $K \approx 5\log(1+z)$, is customarily imposed since a measurement through a single photometric bandpass filter O will record a fraction of the total spectrum that has shifted in red at the observer in relation to an emitted-frame bandpass filter Q (Oke & Sandage 1968; Kim, Goobar & Perlmutter 1996; Hogg et al. 2002). The simple functional form of Eq. 4 for the distance modulus *vs.* redshift $\mu = 5\log(R/r_r) + 2.5\log(z^2(1+z))$ agrees with observations (Fig. 3). It has also been found earlier (Suntola 2009). Since the data analysis by the universal law follows from the conservation of energy, it does not accommodate additional parameters to tailor the fit.

Conservation of energy requires that when mass is combusted to radiation, the change in the kinetic energy $d_t2K = -\partial_tU + \partial_tQ$ will balance the changes in the scalar (gravitational) potential and vector potential, i.e., dissipation. Thus to ensure the balance of energy in the form of matter and radiation, the sources of radiation, i.e., the diverse dissipative mechanisms are on average displacing from each other with increasing average velocity v approaching the speed of light ($v^2/c^2 \to 1$). Notably, if any additional form of energy, say dark, were at play, it would violate the conservation of energy.

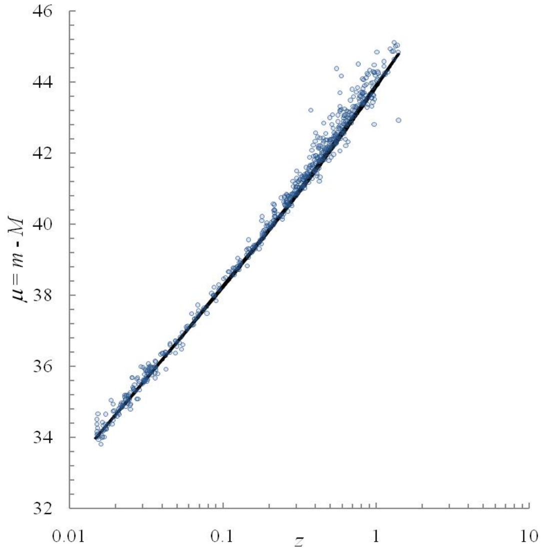

Figure 3. Distance modulus μ vs. redshift z of type 1a supernovae data (http://supernova.lbl.gov/Union/) can be understood according to Eq. 4 (solid line) as a mere manifestation of the least-time propagation of light through the Universe that is expanding at an increasing rate when mass is combusted to radiation. The solid line has been computed from Eq. 4 by setting $R = cT$ where the age of Universe is $T = 13.7\cdot 10^9$ years.

4 DEFLECTION IN REFRACTION

When light passes by a body of mass M_o, not only will its frequency shift but also its momentum will change from that at the sparse universal surroundings to that at the denser local surroundings about the body. Consequently light in a local gravitational potential will curve away from the straight geodesic defined by the universal gravitational potential. The optimal trajectory through a medium of varying index of refraction can be found from the stationary action. The angular momentum of light $L = mr^2\omega = hfr^2\omega/c^2$ does not change when the change df in frequency f balances the change $d\omega$ in the angular frequency $\omega = d_t\phi$ of the angle ϕ between the position vector $\mathbf{r}$ on the trajectory and the reference direction $\mathbf{r}_o$ of the closest approach. In other words, light sweeps equal arcs during equal intervals of time so that $d_\omega f = -f/\omega$ (Fig. 4). Alternatively this Keplerian relation can be stated so that when the radius of curvature r decreases, the frequency will shift blue according to $d_r f = f/r$ and associate with the change in momentum $d_t\mathbf{p}$.

The maximal deflection φ at the closest approach r_o depends on the mass M_o of the local anisotropy in relation to the universal density. Since df balances $d\omega$ on the curved geodesic (Fig. 4), φ can be identified from the stationary condition of angular momentum

$$\begin{aligned} d_t L &= \frac{hf}{c^2} r^2\omega^2 = (2\pi)^2 \frac{v^2}{c^2} hf \\ &= \frac{2\pi^2 GM_\circ}{c^2 r} hf = (2\pi)^2 \frac{M_\circ}{r}\frac{R}{M} hf = \varphi hf \end{aligned} \quad (5)$$

where $\omega = 2\pi/t$ and the kinematic equation $r = ½a_o t^2$ for the local acceleration $a_o = GM_o/r^2$ is used to relate the velocity $v = r/t$ near the mass M_o via the constant of gravitation G. Likewise $R = ½aT^2$ for the universal acceleration $a = GM/R^2$ is used to relate the speed of light $c = R/T$ to the mass M of the Universe that has expanded during time T to the radius R. In this way the maximal angle of deflection that is acquired during light's flyby is found as $\varphi = 2\pi^2 GM_o/c^2 r_o$. The bending can also be described so that the light curves past M_o along the arc $r_o\varphi$ that makes the fraction M_o/M of the universal arc $R\Phi$ (Fig. 4).

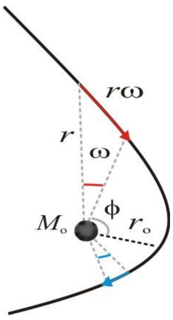

Figure 4. Light curves past an anisotropy of energy density at a minimum distance r_o along the path where equal arcs $r\phi$ (colored) are swept at equal times, i.e., $r\omega$ is a constant.

5 LENSING ABOUT A LOCAL DENSITY

When a body of mass M_o happens to lie exactly along the line that joins us as the observer to the source, an image of a perfect ring will form from a spherical source (Chwolson 1924; Einstein 1936) (Fig. 5). The angular radius $\theta^2 = (2\pi^2 GM_o/c^2)(d_{ls}/d_s d_l)$ is obtained from the lens equation $\theta d_s = \theta_s d_s + \varphi d_{ls}$, where the distance from the observer to the source is d_s and the distance from the observer to the lens is d_l and the distance between the lens and the source is d_{ls}. The perpendicular distance between the light's path and the center of the gravitational field $U(r)$ is the impact parameter $b \approx \theta d_l$.

The lensing mass M_o, such as that of a galaxy, can be determined from the opening angle $\varphi = 2\pi^2 GM_o/c^2 r_o$ of the image cone assuming that the least-time path grazed the lensing body at the minimum distance $r_o \approx b$. It is worth noting that φ obtained from Eq. 5 is about factor of five ($2\pi^2/4 \approx 5$) larger than the corresponding value available from the general relativity (Berry 2001). Thus, according to the principle of least action the gravitational lensing does not provide compelling evidence for large amounts of dark matter in galaxies.

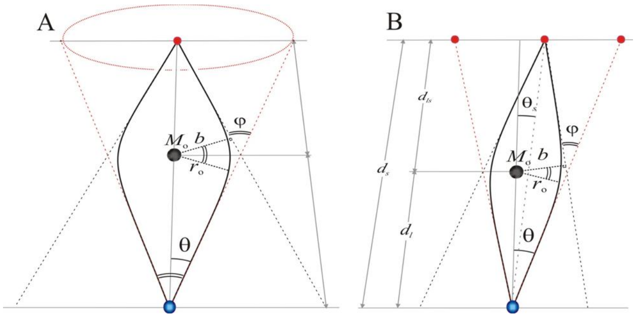

Figure 5. (A) When the source (red) and observer (blue) are located symmetrically about the lensing mass, a ray of light curves about a gravitating body (black) so that the extended lines-of-sight (dashed lines) at the perpendicular distance b from the lens cross at the angle of φ and form an image of a ring at the focal plane. The total deflection φ for the light's flyby at the closest distance $r_o \approx \theta d_l$ is defined relative to the straight geodesic (vertical gray line) in the absence of mass M_o. (B) When the line from the observer to the source deviates from the line to the lens, two or more images may be seen. The distance d_s from the observer to the source is a sum of the distances d_{ls} and d_l from the plane of lens to the source and to the observer.

Conversely, when the lens is not along the line from the observer to the object, two or more images at angles θ_i may form (Walsh, Carswell & Weymann 1979) depending on whether one or more least-time paths of different curvatures are available for the energy dispersal. For example, the lens equation yields for two images separated by $\theta_1 + \theta_2 = (2\pi^2 GM_o/c^2)(d_{ls}/d_s d_l)(\theta_1^{-1} + \theta_2^{-1}) \Rightarrow \theta_1\theta_2 = \theta^2$. Moreover, for a given image the line-of-sight will deviate by an angle $\alpha = \varphi/2 + \beta$ away from the reference geodesic without the lensing mass where β denotes the angle with respect to the symmetry axis of maximal deflection (Fig. 6). Furthermore, when the observer or the source or both are close to the lens, the deflection will be incomplete by an angle γ. Then the image will be seen at the angle $\alpha = \varphi/2 + \beta - \gamma$.

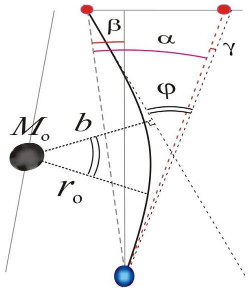

Figure 6. A ray of light from the source (red) curves about a gravitating body of mass M_o (black) to the observer (blue) so that the apparent deflection α falls short from the maximal angle φ by the angle β between the source–observer line (dashed line) and the symmetry axis (gray line) as well as by the angle γ between the extended line-of-sights for the partial and complete optical paths through focal points.

When a ray of starlight grazes the Sun at an eclipse, the line-of-sight will deviate by the angle $\alpha < \varphi/2$ that inclines also by parallax away from the night-sky geodesic (Fig. 7). Since the parallax correction is functionally similar to the bending, the observed deviation is at most $\alpha' < \varphi/4 = \pi^2 GM_o/2c^2 r_o = 2.16''$ according to the elementary analysis that excludes other factors (Froeschle, Mignard & Arenou 1997).

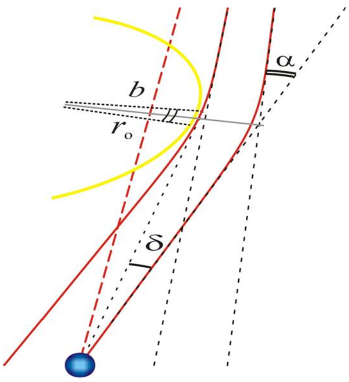

Figure 7. Parallel rays (dashed) from a distant star when grazing the Sun (yellow), will bend by an angle $\alpha < \varphi/2$. The apparent deflection is obtained when the parallax δ is subtracted from α using the night-sky ray (dashed) bound to Earth (blue) as the reference. The parallax is exaggerated for clarity whereas in reality the ensuing offset at the surface of Earth is about 1500 km.

Often a time-delay in signal propagation, when light curves through a local energy density, can be recorded very precisely relative to the propagation along the straight geodesic in the universal gravitational potential (Fig. 6). The delay $\Delta t = r/v$ that accumulates during light's flyby of a body M_o at a distance r with velocity v is found from the least-time condition (Eq. 5). When light grazes the Sun at $r_o\varphi = v\Delta t \approx b\varphi$, the gravitational potential is still modest and the speed of light will slow down only little from c in the universal surroundings. Thus, to a good approximation $\Delta t = 2\pi^2 GM_o/c^3$. This amounts to an excess of 195 μs for a round trip in agreement with radar echo measurements (Shapiro 1964; Shapiro et al. 1968).

In general, the time delay relates the local M_o to the universal mass M via $\Delta t/T = (2\pi)^2 GM_o/2c^2 R = (2\pi)^2 M_o/M$ (Koskela & Annila 2010). For example, the relative time delay Δt between two stationary paths taken by a variable signal (Hewitt 1995) can be used to determine the Hubble constant $H = 1/T = (\Delta\theta)^2 d_L d_S/d_{LS} R\Delta t$ from the separation $\Delta\theta$ of two images (Refsdal 1964). Ultimately, in the limit $M_o \rightarrow M$, the relation yields a full coverage $\Delta\theta = 2\pi$ of the sky by numerous images of nascent Universe. The expansion of Universe contributes to the lensing by a factor $1 + z = R/r_e$ since during the signal propagation the radius of curvature has been increasing from r_e at the time of emission t_e to R at the present time of observation.

6 DISCUSSION

The principle of least time is known to be a powerful way of analyzing propagation of light through a varying energy density. However, the variations in the evolutionary trajectory to be minimized are given by Maupertuis action, whereas the commonly used Lagrangian integrand qualifies only to elucidate paths within stationary surroundings. For this reason the results by Fermat's principle presented here differ from those obtained via general relativity. More generally any formulation that complies with any one group of symmetry, such as that of Poincaré, cannot break its norm which would be necessary to delineate least-time paths through varying energy densities. When the spontaneous symmetry breaking is not understood as a non-unitary process, but the invariant form of the space-time curvature is insisted, the discrepancy between observations of evolutionary processes and predictions will be inevitable. It will prompt one to save the unitary theory by invoking *ad hoc* explanations, most notably dark energy and dark matter or to propose impromptu expansions, most notably modified gravity. In short *We cannot solve problems by using the same kind of thinking we used when we created them* (Calaprice 2005).

Obviously energy density gradients affect not only rays of light but also paths of bodies. To this end the principle of least action á la Maupertuis accounts also for galactic rotational curves and anomalous accelerations as well as for advancing perihelion precession (Koskela & Annila 2010; Annila 2009). Thus the universal principle provides a holistic and self-consistent worldview in an elementary mathematical form (Annila 2010). The Universe is irrevocably processing from high-symmetry states of bound energy to states of lower and lower symmetry, eventually attaining the lowest group U(1). This free form of energy is electromagnetic radiation. In thermodynamic terms it makes sense to measure all bound forms of energy via $E = mc^2 = m/\varepsilon_o\mu_o$ relative to the free space, the lowest state, characterized by permittivity ε_o and permeability μ_o. This is to say that the speed of light is dictated by the surrounding energy density of any kind, most notably, by that of free space. The cosmological principle, i.e., the high degree of homogeneity at the largest scale is, according to the thermodynamic tenet, a mere consequence of maximal dispersal of energy. It is the combustion of bound forms of energy to the free form of energy by stars, pulsars, black holes etc. that powers the expansion. This is to say, the Big Bang did not happen – it is still going on.

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