Computability-Theoretic Categoricity and Scott Families

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Complexity of isomorphisms

Goal for this talk:
We study the complexity of isomorphisms between a computable structure and its countable/computable copies.

Definition
A computable structure $\mathcal{M}$ is **computably categorical** (autostable) if, for every computable structure $\mathcal{A}$ isomorphic to $\mathcal{M}$, there exists a computable isomorphism from $\mathcal{M}$ onto $\mathcal{A}$.

Theorem (Fröhlich and Shepherdson, 1956)
There exist examples of computable fields, extensions of the rationals, of both finite and infinite transcendence degrees, which are not computably categorical.

Theorem (Mal’cev, 1962)
There exist a computable group which is not computably categorical.
Computationally categorical structures

- A vector space is c.c. iff it has a finite basis.
- Goncharov and Dzgoev (1980); Remmel (1981): A linear order is c.c. iff it has only finitely many successor pairs.
- Goncharov and Dzgoev; Remmel; LaRoche (1977): A computable Boolean algebra is c.c. if it has finitely many atoms.
- Goncharov, Lempp and Solomon (2003): ordered abelian groups and ordered Archimedean groups.
- Calvert, Cenzer, Harizanov (2009), and Morozov: equivalence structures.
Relative computable categoricity

If we consider arbitrary countable copies of a computable structure:

**Definition**
A computable structure $\mathcal{M}$ is *relatively computably categorical* if for every $\mathcal{A}$ isomorphic to $\mathcal{M}$, there is an isomorphism from $\mathcal{M}$ to $\mathcal{A}$, which is computable relative to the atomic diagram of $\mathcal{A}$.

Relative computable categoricity implies computable categoricity.

**Theorem (Goncharov, 1977)**
*Computable categoricity of a computable structure does not imply its relative computable categoricity.*
Computably categorical not relatively c.c.

- Hirschfeldt, Khoussainov, Shore, and Slinko: directed graphs, undirected graphs, partial orders, lattices, rings (with zero-divisors), integral domains of arbitrary characteristic, commutative semigroups, and 2-step nilpotent groups
- R. Miller, Park, Poonen, Schoutens, and A. Shlapentokh: fields.
- Hirschfeldt, Kramer, R. Miller, and Shlapentokh characterized relative computable categoricity for computable algebraic fields.

**Theorem (Hirschfeldt, Kramer, R. Miller, and Shlapentokh, 2015)**

There is a computably categorical algebraic field, which is not relatively computably categorical.
When computably categorical is relatively c.c.

For linear orders, Boolean algebras, trees of finite height, abelian $p$-group, equivalence structures, injection structures, algebraic fields with a splitting algorithm, computable categoricity implies relative computable categoricity.
**Definition**

A computable structure $M$ is **d-computably categorical** if, for every computable structure $A$ isomorphic to $M$, there exists a $d$-computable isomorphism from $M$ onto $A$.

In the case when $d = 0^{(n-1)}$, $n \geq 1$, we also say that $M$ is $\Delta_n^0$-categorical. Thus, computably categorical is the same as $0$-computably categorical or $\Delta_1^0$-categorical.

**Example**

The structures of Frohlich-Shepherdson and Malcev are $0'$-categorical ($\Delta_2^0$-categorical).
$\Delta_0^n$-categoricity in familiar classes

- Ash (1986): a complete description of higher levels categoricity for well-orders.
- McCoy (2003): characterized, under certain restrictions, $\Delta_2^0$-categorical linear orders and Boolean algebras.
- Barker (1995): for every computable ordinal $\alpha$, there are $\Delta_{2\alpha+2}^0$-categorical but not $\Delta_{2\alpha+1}^0$-categorical abelian $p$-groups.
- Lempp, McCoy, R. Miller, and Solomon (2005): for every $n \geq 1$, there is a computable tree of finite height, which is $\Delta_{n+1}^0$-categorical but not $\Delta_n^0$-categorical.
Relative $\Delta^0_\alpha$-categoricity

Again, consider arbitrary countable copies of a computable structure:

**Definition**

A computable structure $\mathcal{M}$ is *relatively $\Delta^0_\alpha$-categorical* if for every $\mathcal{A}$ isomorphic to $\mathcal{M}$, there is an isomorphism from $\mathcal{M}$ to $\mathcal{A}$, which is $\Delta^0_\alpha$ relative to the atomic diagram of $\mathcal{A}$.

Clearly, a relatively $\Delta^0_\alpha$-categorical structure is $\Delta^0_\alpha$-categorical.
Scott families

- Scott families come from Scott isomorphism theorem, which says that for a countable structure $\mathcal{A}$, there is an $L_{\omega_1\omega}$-sentence the countable models of which are exactly the isomorphic copies of $\mathcal{A}$.

- A Scott family for a structure $\mathcal{A}$ is a countable family $\Phi$ of $L_{\omega_1\omega}$-formulas with finitely many fixed parameters from $\mathcal{A}$ such that:
  (i) Each finite tuple in $\mathcal{A}$ satisfies some $\psi \in \Phi$; 
  (ii) If $\bar{a}, \bar{b}$ are tuples in $\mathcal{A}$, of the same length, satisfying the same formula in $\Phi$, then there is an automorphism of $\mathcal{A}$, which maps $\bar{a}$ to $\bar{b}$.

- A formally $\Sigma^0_\alpha$ Scott family is a $\Sigma^0_\alpha$ Scott family consisting of computable $\Sigma_\alpha$ formulas. In particular, it follows that a formally c.e. Scott family is a c.e. Scott family consisting of finitary existential formulas.
Syntactic characterization of relative categoricity

Theorem (Goncharov (1975); Ash, Knight, Manasse, and Slaman (1989); Chisholm, 1990)

The following are equivalent for a computable structure $\mathcal{A}$.

1. The structure $\mathcal{A}$ is relatively $\Delta^0_\alpha$-categorical.
2. The structure $\mathcal{A}$ has a formally $\Sigma^0_\alpha$ Scott family $\Phi$.
3. The structure $\mathcal{A}$ has a formally c.e. Scott family $\Phi$ of computable $\Sigma^0_\alpha$ formulas.
Relatively $\Delta^0_n$-categorical structures

- McCoy (2003): A computable linear order is relatively $\Delta^0_2$-categorical if and only if it is a sum of finitely many intervals, each of type $m, \omega, \omega^*, \mathbb{Z}$, or $n \cdot \eta$, so that each interval of type $n \cdot \eta$ has a supremum and infimum.

- McCoy: A computable Boolean algebra is relatively $\Delta^0_2$-categorical if and only if it can be expressed as a finite direct sum $c_1 \lor \cdots \lor c_n$, where each $c_i$ is either atomless, an atom, or a 1-atom.

- Calvert, Cenzer, Harizanov, Morozov (2006): A computable equivalence structure is relatively $\Delta^0_2$-categorical if and only if it either has finitely many infinite equivalence classes, or there is an upper bound on the size its finite equivalence classes.
More examples

• Cenzer, Harizanov, Remmel (2014): A computable injection structure is relatively $\Delta^0_2$-categorical if and only if it has finitely many orbits of type $\omega$, or finitely many orbits of type $\mathbb{Z}$.

• Calvert, Cenzer, Harizanov, and Morozov (2009): A computable abelian $p$-group $G$ is relatively $\Delta^0_2$-categorical iff $G$ is reduced and $\lambda(G) \leq \omega$, or $G$ is isomorphic to $\bigoplus \mathbb{Z}(p^{\infty}) \oplus H$, where $\alpha \leq \omega$ and $H$ has finite period.

• Every computable injection structure is relatively $\Delta^0_3$-categorical.

• Every computable equivalence structure is relatively $\Delta^0_3$-categorical.
\(\Delta^0_\alpha\)-categorical not relatively \(\Delta^0_\alpha\)-categorical

Theorem (Goncharov)

*Computable categoricity of a computable structure does not imply its relative computable categoricity.*

Theorem (Goncharov, Harizanov, Knight, McCoy, R. Miller, and Solomon (2005); Chisholm, Fokina, Goncharov, Harizanov, Knight, and Quinn, 2009)

*For every computable ordinal \(\alpha\), there is a \(\Delta^0_\alpha\)-categorical but not relatively \(\Delta^0_\alpha\)-categorical structure.*

The same holds for directed graphs, undirected graphs, partial orders, fields, etc.
Theorem (Kach and Turetsky, 2009)

There exists a $\Delta^0_2$-categorical equivalence structure, which is not relatively $\Delta^0_2$-categorical.

On the other hand,

- Cenzer, Harizanov, and Remmel (2014): Every $\Delta^0_2$-categorical injection structure is relatively $\Delta^0_2$-categorical.
- Bazhenov (2014): Every $\Delta^0_2$-categorical Boolean algebra is relatively $\Delta^0_2$-categorical.
Theorem (Goncharov, 1975)

A 2-decidable structure is computably categorical if and only if it is relatively computably categorical.

Kudinov (1996): an example of 1-decidable and computably categorical structure, which is not relatively computably categorical.

Theorem (Ash, 1987)

For every computable ordinal $\alpha$, under certain decidability conditions on $A$, if $A$ is $\Delta^0_\alpha$-categorical, then $A$ is relatively $\Delta^0_\alpha$-categorical.
Categoricity and relative categoricity

Theorem (Downey, Kach, Lempp, and Turetsky, 2013)
Any 1-decidable computably categorical structure is relatively \( \Delta^0_2 \)-categorical.

Theorem (Downey, Kach, Lempp, Lewis, Montalbán, and Turetsky, 2015)
For every computable ordinal \( \alpha \), there is a computably categorical structure that is not relatively \( \Delta^0_\alpha \)-categorical.
Fraïssé limits

Definition
A countable structure is a *Fraïssé limit* of a class of finitely generated structures $\mathbb{K}$ if it is ultrahomogeneous, and has age $\mathbb{K}$.

- The *age* of a structure $\mathcal{M}$ is the class of all finitely generated structures that can be embedded in $\mathcal{M}$.
- Fraïssé: a finite or countable class $\mathbb{K}$ of finitely generated structures is the age of a finite or a countable structure iff $\mathbb{K}$ has the hereditary property and the joint embedding property.
- A class $\mathbb{K}$ has the *hereditary property* if whenever $\mathcal{C} \in \mathbb{K}$ and $\mathcal{S}$ is a finitely generated substructure of $\mathcal{C}$, then $\mathcal{S}$ is isomorphic to some structure in $\mathbb{K}$.
- A class $\mathbb{K}$ has the *joint embedding property* if for every $\mathcal{B}, \mathcal{C} \in \mathbb{K}$ there is $\mathcal{D} \in \mathbb{K}$ such that $\mathcal{B}$ and $\mathcal{C}$ embed into $\mathcal{D}$.
- A structure $\mathcal{U}$ is *ultrahomogeneous* if every isomorphism between finitely generated substructures of $\mathcal{U}$ extends to an automorphism of $\mathcal{U}$. 
Categoricity of Fraïssé limits I

Theorem
Let $\mathcal{A}$ be a computable structure which is a Fraïssé limit. Then $\mathcal{A}$ is relatively $\Delta^0_2$-categorical.

Idea of the proof:
Because of ultrahomogeneity, we can construct isomorphisms between $\mathcal{A}$ and an isomorphic structure $\mathcal{B}$ using a back-and-forth argument, as long as we can determine, for every $\bar{a} \in \mathcal{A}$ and $\bar{b} \in \mathcal{B}$, whether there is an isomorphism from the structure generated by $\bar{a}$ to the structure generated by $\bar{b}$ that maps $\bar{a}$ to $\bar{b}$ in order. This can be determined by $(\mathcal{B})'$, since there is such an isomorphism precisely if there is no atomic formula $\varphi$ with $\mathcal{A} \models \varphi(\bar{a})$ and $\mathcal{B} \not\models \varphi(\bar{b})$. This is a $\Pi^0_1$ condition relative to $\mathcal{A} \oplus \mathcal{B} \equiv_T \mathcal{B}$. Therefore, we can use $(\mathcal{B})'$ as an oracle to perform the back-and-forth construction of an isomorphism, and so there is a $\Delta^0_2(\mathcal{B})$ isomorphism.
Categoricity of Fraïssé limits II

Note that if the language of $\mathcal{A}$ is finite and relational, then there are only finitely many atomic formulas $\varphi$ to consider, and the set of such formulas can be effectively determined. Hence, if the language is finite and relational, then a Fraïssé limit is necessarily relatively computably categorical.

**Theorem**

*There is a 1-decidable structure $\mathcal{F}$ that is a Fraïssé limit and computably categorical, but not relatively computably categorical. Moreover, the language for such $\mathcal{F}$ can be finite or relational.*

Idea of the proof: take structures by Kudinov or Downey, Kach, Lempp, Turetsky and make them Fraïssé limits.
Categoricity for Trees

- Lempp, McCoy, R. Miller, and Solomon: characterized computably categorical trees of finite height, and showed that for these structures, computable categoricity coincides with relative computable categoricity.
- Lempp, McCoy, R. Miller, and Solomon: for every $n \geq 1$, there is a computable tree of finite height, which is $\Delta^0_{n+1}$-categorical but not $\Delta^0_n$-categorical.
- There is no known characterization of $\Delta^0_2$-categoricity or higher level categoricity for trees of finite height.
Theorem

There is a computable $\Delta^0_2$-categorical tree of finite height, which is not relatively $\Delta^0_2$-categorical.
Theorem
There is a computable $\Delta^0_2$-categorical tree of finite height, which is not relatively $\Delta^0_2$-categorical.
$R_i : \mathcal{X}_i$ with parameters $\bar{p}_i$ is not a Scott family for $\mathcal{T}$.

If $\varphi(\bar{x})$ is a computable $\Sigma_2$ formula and $\bar{a} \in \mathcal{T}$, then $\mathcal{T} \models \varphi(\bar{a})$ if and only if $\mathcal{T}_s \models \varphi(\bar{a})$ for co-finitely many stages $s$.

$Q_j :$ If $M_j \simeq \mathcal{T}$, then there is a $0'$-computable isomorphism between $M_j$ and $\mathcal{T}$. 
$R_i : \mathcal{X}_i$ with parameters $\bar{p}_i$ is not a Scott family for $\mathcal{T}$.

If $\varphi(\bar{x})$ is a computable $\Sigma_2$ formula and $\bar{a} \in \mathcal{T}$, then $\mathcal{T} \models \varphi(\bar{a})$ if and only if $\mathcal{T}_s \models \varphi(\bar{a})$ for co-finitely many stages $s$.

$Q_j :$ If $M_j \cong \mathcal{T}$, then there is a $0'$-computable isomorphism between $M_j$ and $\mathcal{T}$.
$R_i : \mathcal{X}_i$ with parameters $\overline{p}_i$ is not a Scott family for $\mathcal{T}$.

If $\varphi(\overline{x})$ is a computable $\Sigma_2$ formula and $\overline{a} \in \mathcal{T}$, then $\mathcal{T} \models \varphi(\overline{a})$ if and only if $\mathcal{T}_s \models \varphi(\overline{a})$ for co-finitely many stages $s$.

$Q_j :$ If $M_j \cong \mathcal{T}$, then there is a $0'$-computable isomorphism between $M_j$ and $\mathcal{T}$.
$R_i : \mathcal{X}_i$ with parameters $\bar{p}_i$ is not a Scott family for $\mathcal{T}$.

If $\varphi(\bar{x})$ is a computable $\Sigma_2$ formula and $\bar{a} \in \mathcal{T}$, then $\mathcal{T} \models \varphi(\bar{a})$ if and only if $\mathcal{T}_s \models \varphi(\bar{a})$ for co-finitely many stages $s$.

$Q_j :$ If $M_j \cong \mathcal{T}$, then there is a $0'$-computable isomorphism between $M_j$ and $\mathcal{T}$. 
R. Miller established that no computable tree of infinite height is computably categorical.

**Theorem**

There is a computable $\Delta^0_2$-categorical tree of infinite height, which is not relatively $\Delta^0_2$-categorical.
Abelian $p$-groups

- A group $G$ is called a $p$-group if for all $g \in G$, the order of $g$ is a power of $p$.
- Barker: for every computable ordinal $\alpha$, there is a $\Delta^0_{2\alpha+2}$-categorical but not $\Delta^0_{2\alpha+1}$-categorical abelian $p$-group.
- $\mathbb{Z}(p^n)$ is the cyclic group of order $p^n$.
- $\mathbb{Z}(p^\infty)$ is the quasicyclic (Prüfer) abelian $p$-group, the direct limit of the sequence $\mathbb{Z}(p^n)$.
- Goncharov; Smith: computably categorical abelian $p$-groups are those that can be written as:
  (i) $\bigoplus_{l \leq \omega} \mathbb{Z}(p^\infty) \oplus F$ for $l \leq \omega$ and $F$ is a finite group; or
  (ii) $\bigoplus_{n} \mathbb{Z}(p^\infty) \oplus H \oplus \bigoplus_{\omega} \mathbb{Z}(p^k)$, where $n, k \in \omega$ and $H$ is a finite group.
- For these groups, computable categoricity and relative computable categoricity coincide.
Categoricity for abelian $p$-groups

Calvert, Cenzer, Harizanov, and Morozov (2009): a computable abelian $p$-group $G$ is relatively $\Delta^0_2$-categorical if and only if:

(i) $G$ is isomorphic to $\bigoplus_l \mathbb{Z}(p^\infty) \oplus H$, where $l \leq \omega$ and $H$ has finite period; or

(ii) All elements in $G$ are of finite height (equivalently, $G$ is reduced with $\lambda(G) \leq \omega$).

Theorem

There is a computable $\Delta^0_2$-categorical abelian $p$-group, which is not relatively $\Delta^0_2$-categorical.

Idea of the proof:

Define

$$G = \bigoplus_{\omega} \mathbb{Z}(p^\infty) \oplus \bigoplus_{k \in A} \mathbb{Z}(p^k),$$

where $A$ is a specific $\Delta^0_2$ set.
Torsion free groups

Definition

A homogenous completely decomposable abelian group is a group of the form \( \bigoplus_{i \in \kappa} H \), where \( H \) is a subgroup of the additive group of the rationals, \((\mathbb{Q}, +)\).

- A HCDAP is computably categorical iff \( \kappa \) is finite.
- For \( P \) a set of primes, define \( Q^{(P)} \) to be the subgroup of \((\mathbb{Q}, +)\) generated by \( \{ \frac{1}{p^k} : p \in P \land k \in \omega \} \).

Theorem (Downey and Melnikov, 2013)

A computable homogenous completely decomposable abelian group of infinite rank is \( \Delta^0_2 \)-categorical if and only if it is isomorphic to \( \bigoplus Q^{(P)} \), where \( P \) is c.e. and the set \((\text{Primes} – P)\) is semi-low.

Recall: a set \( S \subseteq \omega \) is semi-low if the set \( H_S = \{ e : W_e \cap S \neq \emptyset \} \) is computable from \( \emptyset' \).
Categoricity for HCDAGs

Theorem
A computable homogenous completely decomposable abelian group of infinite rank is relatively $\Delta^0_2$-categorical if and only if it is isomorphic to $\bigoplus_{\omega} Q^{(P)}$, where $P$ is a computable set of primes.

Idea of proof:
By Downey-Melnikov, $P$ is c.e. Then we prove that $P$ is also co-c.e.

Corollary
There is a computable homogenous completely decomposable abelian group, which is $\Delta^0_2$-categorical but not relatively $\Delta^0_2$-categorical.
Let $\mathcal{M}$ be any computable structure.

**Definition**

The **categoricity spectrum** of $\mathcal{M}$ is the set

$$CatSpec(\mathcal{M}) = \{ d | \mathcal{M} \text{ is } d\text{-computably categorical} \}.$$ 

A Turing degree $d$ is the **degree of categoricity** of $\mathcal{M}$ if $d$ is the least degree in $CatSpec(\mathcal{M})$, if it exists.
Theorem (F., Kalimullin, R. Miller, 2010)

For $n \in \omega$, each degree d-c.e. in and above $0^{(n)}$ is the degree of categoricity of a computable structure.

- The result was lifted to hyperarithmetical levels by Csima, Franklin and Shore.
- The result is also true for directed graphs, undirected graphs, partial orders, fields, etc...
Degrees of categoricity for equivalence structures

Easily follows from Calvert, Cenzer, Harizanov, and Morozov (2006):

**Theorem**

*The only degrees of categoricity for relatively $\Delta^0_2$-categorical equivalence structures are $0$ and $0'$.*

Barbara Csima and Keng Meng Ng recently announced that the only possible degrees of categoricity for equivalence structures can be $0, 0', 0''$. 
Degrees of categoricity II

Theorem (Cenzer, Harizanov, and Remmel)

*The degrees of categoricity of computable injections structures can only be \(0, 0'\) and \(0''\).*

Theorem (Frolov, 2015)

- *The degrees of categoricity of relatively \(\Delta^0_2\)-categorical linear orders can only be \(0\) and \(0'\).*
- *For every \(d \geq 0''\), \(d\) is the degree of categoricity of a linear order.*

Theorem (Bazhenov, 2014)

*The degrees of categoricity of relatively \(\Delta^0_2\)-categorical (equivalently, \(\Delta^0_2\)-categorical) Boolean algebras can only be \(0\) and \(0'\).*
Degrees of categoricity for Boolean algebras

Theorem
The degrees of categoricity of relatively $\Delta_3^0$-categorical Boolean algebras can only be 0, 0’ and 0”.

Use McCoy’s characterization of relatively $\Delta_3^0$-categorical Boolean algebras:

Theorem (McCoy)
A Boolean algebra is relatively $\Delta_3^0$-categorical iff it can be expressed as finite direct sums of algebras that are atoms, atomless, 1-atoms, rank 1 atomic, or isomorphic to the interval algebra $I(\omega + \eta)$. 
Boolean algebras

- A Boolean algebra $B$ is *atomic* if for every $a \in B$ there is an atom $b \leq a$. An equivalence relation $\sim$ on a Boolean algebra $A$ is defined by:

  \[
  a \sim b \text{ iff each of } a \cap \overline{b} \text{ and } b \cap \overline{a} \text{ is } \emptyset \text{ or a union of finitely many atoms of } A.
  \]

- A Boolean algebra $A$ is a *1-atom* if $A/\sim$ is a two-element algebra.

- A Boolean algebra $A$ is *rank 1* if $A/\sim$ is a nontrivial atomless Boolean algebra.

- McCoy: a countable rank 1 atomic Boolean algebra is isomorphic to $I(2 \cdot \eta)$. 
Proof sketch

Theorem (McCoy)

• A Boolean algebra is relatively $\Delta^0_3$-categorical iff it can be expressed as finite direct sums of algebras that are atoms, atomless, 1-atoms, rank 1 atomic, or isomorphic to the interval algebra $I(\omega + \eta)$.

• Fix a relatively $\Delta^0_3$-categorical Boolean algebra $B$. Assume that $B$ has a summand which is either rank 1 atomic or isomorphic to the interval algebra $I(\omega + \eta)$.

• All of the potential summands in the characterization of relatively $\Delta^0_3$-categorical Boolean algebras have computable isomorphic copies in which the set of finite elements is computable.

• We show that both the rank 1 atomic algebra and $I(\omega + \eta)$ have computable isomorphic copies where the set of finite elements is $\Sigma^0_2$-complete.
Proof sketch

• Let $\mathcal{C}$ be a computable copy of $I(\omega + \eta)$ in which the set of atoms $\{a_i : i \in \omega\}$ is computable.

• Let $\varphi(i, x)$ be a computable formula such that

$$i \in \emptyset'' \iff \exists^{<\infty} x \varphi(i, x).$$

• If $\varphi(i, s)$ holds:

$$a_{2i}$$
Proof sketch

- Let $\mathcal{C}$ be a computable copy of $l(\omega + \eta)$ in which the set of atoms $\{a_i : i \in \omega\}$ is computable.
- Let $\varphi(i, x)$ be a computable formula such that
  \[ i \in \emptyset'' \iff \exists^{<\infty} x \varphi(i, x). \]
- If $\varphi(i, s)$ holds:

\[
\begin{array}{c}
\ a_{2i} \\
/ \ \\
| \\
/ \ \\
b_i^1 \hspace{1cm} b_i^0
\end{array}
\]
Proof sketch

• Let $C$ be a computable copy of $I(\omega + \eta)$ in which the set of atoms $\{a_i : i \in \omega\}$ is computable.

• Let $\varphi(i, x)$ be a computable formula such that

$$i \in 0'' \iff \exists^{<\infty} x \varphi(i, x).$$

• If $\varphi(i, s)$ holds:

```
    a_{2i}
   /   \
  b_i^1  b_i^0
 /   \
/b_i b_i
```

```
  b_i^{11}  b_i^{10}
 /   \
/b_i b_i
```

```
  b_i^{01}  b_i^{00}
 /   \
/b_i b_i
```
Proof sketch

- Let $\mathcal{C}$ be a computable copy of $l(\omega + \eta)$ in which the set of atoms $\{a_i : i \in \omega\}$ is computable.
- Let $\varphi(i, x)$ be a computable formula such that
  \[ i \in \emptyset'' \iff \exists^{<\infty} x \varphi(i, x). \]
- If $\varphi(i, s)$ holds:

```
  a_{2i}
    /   \  \
  b^1_i / \  \ b^0_i
    /  \ /  \ /  \
  b^{11}_i b^{10}_i b^{01}_i b^{00}_i
    /  \  /  \  /  \
  b^{111}_i b^{110}_i b^{101}_i b^{100}_i b^{011}_i b^{010}_i b^{001}_i b^{000}_i
```
Degrees of categoricity for $p$-groups

Recall:

Theorem (Calvert, Cenzer, Harizanov, and Morozov)

A computable abelian $p$-group $G$ is relatively $\Delta_2^0$-categorical if and only if:

- $G \cong \bigoplus_{l} \mathbb{Z}(p^\infty) \oplus H$, where $l \leq \omega$ and $H$ has finite period; or
- All elements in $G$ are of finite height.

Proposition

The categoricity degrees of computable relatively $\Delta_2^0$-categorical abelian $p$-groups can only be $0$ and $0'$. 
Open questions

- Do $\Delta^0_2$-categoricity and relative $\Delta^0_2$-categoricity for linear orders coincide?
- Do $\Delta^1_1$-categoricity and relative $\Delta^1_1$-categoricity (for arbitrary structures) coincide?
Open questions

- Do $\Delta_2^0$-categoricity and relative $\Delta_2^0$-categoricity for linear orders coincide?
- Do $\Delta_1^1$-categoricity and relative $\Delta_1^1$-categoricity (for arbitrary structures) coincide?

Thank you!