An introduction to geometric calculus and its application to electrodynamics

Terje G. Vold[a)]
Department of Physics and Astronomy, Swarthmore College, Swarthmore, Pennsylvania 19081

(Received 8 September 1992; accepted 16 October 1992)

A tutorial of geometric calculus is presented as a continuation of the development of geometric algebra in a previous paper. The geometric derivative is defined in a natural way that maintains the close correspondence between geometric algebra and the algebra of real numbers. The use of geometric calculus in physics is illustrated by expressing some basic results of electrodynamics.

TABLE OF CONTENTS

I. INTRODUCTION

This paper continues the tutorial begun in the previous paper,[1] "An introduction to geometric algebra...," by motivating and constructing a calculus based upon geometric algebra. The style follows that of the previous paper, emphasizing motives and physical and geometrical meaning rather than mathematical conciseness and rigor. A more general but condensed treatment is given by Hestenes;[2] a much more thorough and mathematically rigorous treatment is developed in Hestenes and Sobczyk.[3]

The resulting geometric calculus is a generalization of the conventional calculus of real numbers. The calculus of real numbers, along with vector calculus and complex analysis, are natural subsystems of geometric calculus. Just as the use of complex analysis allows many results involving only real numbers to be more easily proved and understood, so the use of geometric calculus allows many results involving only vectors to be more easily proved and understood.

After development of this calculus in the next four sections, it is applied to electrodynamics. The content of Maxwell's equations written using vector calculus is expressed as a single equation in geometric calculus. Plane wave solutions for the free fields are examined. The general solution for arbitrary charge sources is derived and compared with the conventional expressions using vector calculus. (See Jancewicz[4] for an alternative and more extensive development of electrodynamics using geometric algebra.)

The previous paper developed an algebra of mathematical objects—multivectors—having operations of addition and multiplication. This geometric algebra is very similar to the algebra of the real numbers except that the geometric product between multivectors is generally not commutative. It will therefore be a fairly simple matter to define a calculus for this algebra.

Multivector-valued functions of real-valued arguments present no problem: Standard ideas of real analysis apply with no significant change. The resulting calculus is a powerful system that can be applied very productively to classical mechanics, as illustrated in the previous article and as detailed elsewhere.[5]

The most general kind of derivative possible using geometric algebra—a derivative of a multivector-valued function with respect to multivector-valued argument—results in a very rich system, also detailed elsewhere.[3] Rather than look at this most general case, we will restrict our attention to multivector-valued functions of a vector argument. We can easily think about and visualize a function of position in space and represent the position by a vector, so functions of vector arguments are a good place to start studying geometric calculus. Also, only these are needed for application to electrodynamics.

Our notational conventions will be as follows. Arbitrary multivectors will be denoted with nonbold characters without ornamentation such as F. If we wish to identify the grade of a simple multivector, we will use no ornamentation to denote a scalar such as the magnitude B of the magnetic field, boldface type to identify a vector such as the magnetic vector field $\mathbf{B}$, a curved line overscript to identify a bivector such as the magnetic bivector field $\frown{B}$, and a tilde to identify a trivector such as a volume $\tilde{\tau}$. The *unit* trivector is used so often that we will use the special simpler notation of an italic i. Unit vectors will be identified with the usual carrot overscript, such as $\hat{\mathbf{r}}$.

II. DEFINITIONS OF INTEGRALS AND DERIVATIVES

We define an integral in the usual, common-sense way as the limit of a sum (a Reimann integral). For example, consider a multivector-valued function f of a vector argument $\mathbf{r}$, with $\mathbf{r}$ representing position in space. The surface integral of f over the boundary or surface σ of some volume is then defined by

 © 1993 American Association of Physics Teachers

$$\int_\sigma d\hat{\sigma} f(\mathbf{r}) \equiv \lim_{\substack{N\to\infty \\ \Delta\hat{\sigma}_j \to 0}} \sum_{j=1}^{N} \Delta\hat{\sigma}_j f(\mathbf{r}_j), \tag{1}$$

where the sum is over a set of small bivectors $\Delta\hat{\sigma}_j$ that approximate the surface.

The derivative is defined as a generalization of the standard definition in real analysis. For example, for a function $f(t,x)$ that depends on time t and possibly other variables x we define the partial derivative with respect to time as

$$\partial_t f(t,x) \equiv \lim_{\epsilon\to 0} \frac{f(t+\epsilon,x) - f(t,x)}{\epsilon}. \tag{2}$$

We note that this definition has the structure

$$\partial_t f(t,x) = \lim_{\text{interval}\to 0} \frac{(\text{boundary direction}) f(\text{one boundary}) + (\text{boundary direction}) f(\text{other boundary})}{\text{interval}}, \tag{3}$$

where "boundary direction" is either $+1$ or -1. We generalize this to derivatives with respect to a vector argument by (a) replacing the interval of the real number line with a volume represented by a trivector $\tilde{\tau}$, (b) replacing the sum over the two endpoints of the interval with the integral over the volume's boundary or surface, and (c) replacing the boundary directions with bivectors representing oriented segments of the volume's surface, infinitesimal bivectors $d\hat{\sigma}$ in the integral or small but nonzero bivectors $\Delta\hat{\sigma}_j$ in the sum. The result is *the geometric derivative*,[2,3]

$$\nabla f(\mathbf{r}) \equiv \lim_{\tilde{\tau}\to 0} \frac{1}{\tilde{\tau}} \int_\sigma d\hat{\sigma} f(\mathbf{r})$$
$$= \lim_{\tilde{\tau}\to 0} \frac{1}{\tilde{\tau}} \left(\lim_{\substack{\Delta\hat{\sigma}_j\to 0 \\ N\to\infty}} \sum_{j=1}^{N} \Delta\hat{\sigma}_j f(\mathbf{r}_j) \right). \tag{4}$$

This definition may be applied to any multivector-valued function $f(\mathbf{r})$ of a vector argument $\mathbf{r}$: f may be a scalar, vector, bivector, trivector, or any sum of these.

In practice it is very helpful to choose the volume to be a very small cube (sometimes other simple shapes are convenient). The sum in (4) may then be approximated by a sum over the six faces of the cube described by bivectors $\Delta\hat{\sigma}_j$ with centers at positions $\mathbf{r}_j$:

$$\nabla f(\mathbf{r}) \approx \frac{1}{\tilde{\tau}} \sum_{j=1}^{6} \Delta\hat{\sigma}_j f(\mathbf{r}_j). \tag{5}$$

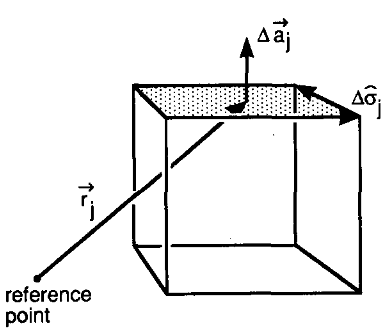

Fig. 1. A cube used in the definition of the geometric derivative. The significant quantities associated with face j are labeled: the vector $\mathbf{r}_j$ describing the position of the center of the face, and the tangential bivector $\Delta\hat{\sigma}_j$ and normal vector $\Delta\mathbf{a}_j \equiv -i\Delta\hat{\sigma}_j$ describing the size and orientation of the face.

See Fig. 1. Of course δ–ϵ arguments can be used to show that this approximation gets as good as we want if we take a small enough volume τ, but we will not worry about such rigor here.

We will now rewrite (5) in a form that is easier to visualize and interpret. We are free to associate the factors on the right side of (5) any way we like, including

$$\nabla f(\mathbf{r}) \approx \sum_{j=1}^{6} \left(\frac{1}{\tilde{\tau}} \Delta\hat{\sigma}_j \right) f(\mathbf{r}_j). \tag{6}$$

The volume $\tilde{\tau}$ is a trivector, so it equals its magnitude τ, a scalar, times the unit trivector i:

$$\tilde{\tau} = i|\tilde{\tau}| = i\tau. \tag{7}$$

We define vectors $\Delta\mathbf{a}_j$ to be perpendicular to the corresponding bivectors $\Delta\hat{\sigma}_j$ and with equal magnitude, so that the $\Delta\mathbf{a}_j$ are associated with small areas as is commonly done in conventional vector analysis. Using result f) of Sec. VI of the preceding paper,[1] these vectors and bivectors are related by

$$i\Delta\mathbf{a}_j = \Delta\hat{\sigma}_j. \tag{8}$$

Substituting these expressions into the last equation for the derivative and canceling the factor of i, we have

$$\nabla f(\mathbf{r}) \approx \sum_{j=1}^{6} \left(\frac{1}{\tau} \Delta\mathbf{a}_j \right) f(\mathbf{r}_j) = \frac{1}{\tau} \sum_{j=1}^{6} \Delta\mathbf{a}_j f(\mathbf{r}_j). \tag{9}$$

III. EVEN AND ODD PARTS OF THE GEOMETRIC DERIVATIVE

This last expression, Eq. (9), gives the geometric derivative as simply the sum of six simple, similar products. As far as algebraic properties go, the symbol "∇" must behave just like any one of the vectors $\Delta\mathbf{a}_j$ since according to (9) it corresponds to these vector factors in the sum. In particular, the geometric derivative can be divided into parts of opposite symmetry,

$$\nabla f(\mathbf{r}) = \nabla \cdot f(\mathbf{r}) + \nabla \wedge f(\mathbf{r}), \tag{10}$$

where

$$\nabla \cdot f(\mathbf{r}) \approx \sum_{j=1}^{6} \left(\frac{1}{\tilde{\tau}} \Delta\hat{\sigma}_j \right) \cdot f(\mathbf{r}_j) = \frac{1}{\tau} \sum_{j=1}^{6} \Delta\mathbf{a}_j \cdot f(\mathbf{r}_j) \tag{11}$$

and

$$\nabla \wedge f(\mathbf{r}) \approx \sum_{j=1}^{6} \left(\frac{1}{\tilde{\tau}} \Delta \hat{\sigma}_j \right) \wedge f(\mathbf{r}_j) = \frac{1}{\tau} \sum_{j=1}^{6} \Delta \mathbf{a}_j \wedge f(\mathbf{r}_j). \tag{12}$$

These expressions depend on the grade of f. If f is a *scalar*-valued function of a vector, the inner product is zero and the outer product is just an ordinary product giving the conventional *gradient* $\nabla f(\mathbf{r})$. If f is a *vector*-valued function $\mathbf{f}$, the inner differential product $\nabla \cdot \mathbf{f}(\mathbf{r})$ is the conventional *divergence*. The outer differential product $\nabla \wedge \mathbf{f}(\mathbf{r})$ is in this case simply related to the *curl*, $\nabla \times \mathbf{f}(\mathbf{r})$, by

$$\nabla \wedge \mathbf{f} = i(\nabla \times \mathbf{f}), \tag{13}$$

just as the corresponding nondifferential vector products are related according to result g) of Sec. VI in the preceding paper.[1] Multiplying this by $-i$ we have

$$\nabla \times \mathbf{f} = -i(\nabla \wedge \mathbf{f}). \tag{14}$$

Then

$$\nabla \times \mathbf{f} = -i[\nabla \wedge \mathbf{f}(\mathbf{r})]$$
$$\approx -i \frac{1}{\tau} \sum_{j=1}^{6} \Delta \mathbf{a}_j \wedge \mathbf{f}(\mathbf{r}_j) = \frac{1}{\tau} \sum_{j=1}^{6} \Delta \mathbf{a}_j \times \mathbf{f}(\mathbf{r}_j). \tag{15}$$

A more familiar-looking expression of the curl may be derived:

$$\nabla \times \mathbf{f} = -i[\nabla \wedge \mathbf{f}(\mathbf{r})]$$
$$\approx -i \sum_{j=1}^{6} \left(\frac{1}{i\tau} \Delta \hat{\sigma}_j \right) \wedge \mathbf{f}(\mathbf{r}_j)$$
$$= -\frac{1}{\tau} \sum_{j=1}^{6} \Delta \hat{\sigma}_j \cdot \mathbf{f}(\mathbf{r}_j) = +\frac{1}{\tau} \sum_{j=1}^{6} \mathbf{f}(\mathbf{r}_j) \cdot \Delta \hat{\sigma}_j. \tag{16}$$

The last expression of (16) corresponds to the usual coordinate-free definition of the curl in which the inner product of the curl with an arbitrary unit vector is defined in terms of a closed line integral $\oint \mathbf{f}(\mathbf{r}) \cdot d\boldsymbol{\lambda} \approx \Sigma_j \mathbf{f}(\mathbf{r}_j) \cdot d\boldsymbol{\lambda}_j$ of an inner product between $\mathbf{f}$ and vector line elements $d\boldsymbol{\lambda}_j$. The inner products in (16) between the vector field $\mathbf{f}$ and bivectors $\Delta \hat{\sigma}_j$ result in vectors in the correct direction and with magnitudes proportional to $\mathbf{f}(\mathbf{r}_j) \cdot d\boldsymbol{\lambda}_j$, and the surface integral takes the place of three independent loop integrals. This correspondence can be made more explicit by taking the inner product of Eq. (16) with any unit vector and considering a pancakelike volume τ perpendicular to the unit vector: By construction, the faces of the pancake do not contribute to the inner product, and the inner product with the surface integral around the perimeter becomes a line integral in the limit of a very thin pancake.

IV. THE PRODUCT AND CHAIN RULES

The product rule for evaluating the derivative $\nabla(fg)$ of a product of functions $f(\mathbf{r})$ and $g(\mathbf{r})$ is a straightforward generalization of the corresponding rule in conventional vector calculus:

$$\nabla(fg) = \dot{\nabla} \dot{f} g + \dot{\nabla} f \dot{g}. \tag{17}$$

The overdots indicate the function being differentiated by ∇ while the order of symbols is maintained because of the noncommutivity of the geometric product.

A proof of this and clarification of what is meant by the overdots follows immediately by considering the integral of $[f(\mathbf{r}) - f(\mathbf{r}_0)][g(\mathbf{r}) - g(\mathbf{r}_0)]/\tilde{\tau}$ over the surface σ of a small volume $\tilde{\tau}$, where $\mathbf{r}_0$ is any fixed point inside the volume:

$$\frac{1}{\tilde{\tau}} \int_\sigma d\hat{\sigma}(f - f_0)(g - g_0)$$
$$= \frac{1}{\tilde{\tau}} \int_\sigma d\hat{\sigma} f g - \frac{1}{\tilde{\tau}} \int_\sigma d\hat{\sigma} f g_0 - \frac{1}{\tilde{\tau}} \int_\sigma d\hat{\sigma} f_0 g$$
$$+ \frac{1}{\tilde{\tau}} \int_\sigma d\hat{\sigma} f_0 g_0. \tag{18}$$

The last integral is identically zero since $f_0 g_0 \equiv f(\mathbf{r}_0) g(\mathbf{r}_0)$ is a constant and any integral $\int d\hat{\sigma}$ over a closed surface vanishes. The integral on the left-hand side of (18) vanishes in the limit $\tau \to 0$: We let the linear dimension of the volume be l so that $\tau \sim l^3$, $\int d\sigma \sim l^2$, $|g - g_0| \sim l$, and $|f - f_0| \sim l$. Then regardless of the grades of f and g, the magnitude of the integral on the left goes like l and so vanishes in the limit $\tau \to 0$.

Taking this limit then gives

$$\lim_{\tilde{\tau} \to 0} \left(\frac{1}{\tilde{\tau}} \int_\sigma d\hat{\sigma} f g \right) = \lim_{\tilde{\tau} \to 0} \left(\frac{1}{\tilde{\tau}} \int_\sigma d\hat{\sigma} f g_0 \right) + \lim_{\tilde{\tau} \to 0} \left(\frac{1}{\tilde{\tau}} \int_\sigma d\hat{\sigma} f_0 g \right). \tag{19}$$

This is what is meant by (17). We could factor the constant g_0 out of the parenthesis in the first right-hand term, but the f_0 cannot be factored out of the second term unless it is a scalar because of the noncommutivity of the products.

If $f(\mathbf{r})$ is a scalar it commutes with everything and we needn't worry about its order. In that case the product rule simplifies to

$$\nabla(fg) = (\nabla f) g + f(\nabla g). \tag{20}$$

This will be used in the application to electrodynamics.

Because of the noncommutivity of factors, variations of the chain rule are possible. One that will be used in electrodynamics is

$$\dot{f} \dot{\nabla} \dot{g} = \dot{f} \dot{\nabla} g + f \dot{\nabla} \dot{g}; \tag{21}$$

that is,

$$\lim_{\tilde{\tau} \to 0} \left(\int_\sigma f \frac{d\hat{\sigma}}{\tilde{\tau}} g \right) = \lim_{\tilde{\tau} \to 0} \left(\int_\sigma f \frac{d\hat{\sigma}}{\tilde{\tau}} \right) g_0 + f_0 \lim_{\tilde{\tau} \to 0} \left(\int_\sigma \frac{d\hat{\sigma}}{\tilde{\tau}} g \right). \tag{22}$$

The chain rule for the composite of a multivector-valued function $f(\mathbf{g})$ with a vector-valued function $\mathbf{g}(\mathbf{r})$ of a vector argument $\mathbf{r}$ is

$$\nabla_r \{ f[\mathbf{g}(\mathbf{r})] \} = \nabla_r \mathbf{g}(\mathbf{r}) \cdot \nabla_g f(\mathbf{g}), \tag{23}$$

where the subscript on the nabla ∇ indicates the variable with respect to which differentiation is performed. The notational convention for any product between three or more factors and involving inner or outer products, such as the right side of (23), is that the inner and outer products have precedence over the geometric product if paranthesis are not explicitly used. [Recall that although the geometric product is associative, the inner and outer products are *not* separately associative and so expressions like (23) are ambiguous without some such convention.]

Equation (23) is easily proved by considering the approximate linear expansions

$$f(\mathbf{g}+\mathbf{h})=f(\mathbf{g})+\mathbf{h}\cdot\nabla_g f(\mathbf{g}), \tag{24}$$

$$\mathbf{g}(\mathbf{r}+\mathbf{s})=\mathbf{g}(\mathbf{r})+\mathbf{s}\cdot\nabla_r \mathbf{g}(\mathbf{r}), \tag{25}$$

and

$$f[\mathbf{g}(\mathbf{r}+\mathbf{s})]\approx f[\mathbf{g}(\mathbf{r})]+\mathbf{s}\cdot\nabla_r f[\mathbf{g}(\mathbf{r})], \tag{26}$$

true for any small $\mathbf{h}$ and $\mathbf{s}$ as is clear by considering the definition of the derivative. Substituting (25) into (26) and then applying (24) with $\mathbf{h}=\mathbf{s}\cdot\nabla_r g(\mathbf{r})$ gives

$$\mathbf{s}\cdot\nabla_r\{f[\mathbf{g}(\mathbf{r})]\}=[(\mathbf{s}\cdot\nabla_r)\mathbf{g}(\mathbf{r})]\cdot\nabla_g f[\mathbf{g}(\mathbf{r})]. \tag{27}$$

Since this is true for arbitrary $\mathbf{s}$, an expansion of $\nabla_r f$ in terms of three orthonormal vectors implies (23).

The chain rule for the composite of a multivector-valued function f of a scalar valued function g of a vector argument $\mathbf{r}$ is

$$\nabla_r\{f[g(\mathbf{r})]\}=[\nabla_r g(\mathbf{r})]\left(\frac{df}{dg}\right). \tag{28}$$

This is similarly proved using

$$f(g+h)\approx f(g)+h\frac{d}{dg}f(g), \tag{29}$$

in place of (24).

V. THE DERIVATIVES OF SOME ELEMENTARY FUNCTIONS

The derivatives of a few simple functions combined with the product and chain rules allow the evaluation of many more complex derivatives. In this section we evaluate the derivatives of a few very simple functions involving the position vector $\mathbf{r}$ and any constant vector $\mathbf{a}$.

The simplest function of a vector variable $\mathbf{r}$ is the identity function, $\mathbf{r}$ itself. Its geometric derivative is

$$\nabla\mathbf{r}=3. \tag{30}$$

As a geometric product between two vectors, this quantity corresponds, in general, to a sum of a scalar and a bivector. But in this special case the bivector portion vanishes. This may be seen by considering the derivative as defined by the sum (4) over the faces of a small cube. We let the cube have volume $|\tilde{\tau}|=l^3$ for some scalar length l and have faces of area $|\Delta\mathbf{a}_j|=l^2$, and write the positions of the centers of the faces as $\mathbf{r}_j=\mathbf{r}_0+\Delta\mathbf{a}_j/2l$ where $\mathbf{r}_0$ is the position of the center of the cube. Each of the six terms of the sum then has a scalar part of 1/2 while the bivector parts sum to zero. Equation (30) immediately follows. The scalar and bivector parts of (30) are

$$\nabla\cdot\mathbf{r}=3. \tag{31}$$

and

$$\nabla\wedge\mathbf{r}=0. \tag{32}$$

Perhaps the next most complicated function of position $\mathbf{r}$ is the geometric product $\mathbf{ra}$ of $\mathbf{r}$ with a constant vector $\mathbf{a}$. By associativity we have the derivative

$$\nabla(\mathbf{ra})=(\nabla\mathbf{r})\mathbf{a}=3\mathbf{a}. \tag{33}$$

Equation (25) used to prove the chain rule is exact for the identity function $\mathbf{g}(\mathbf{r})=\mathbf{r}$. Setting $\mathbf{g}(\mathbf{r})=\mathbf{r}$ and $\mathbf{s}=\mathbf{a}$ in (25), we have the derivative

$$(\mathbf{a}\cdot\nabla)\mathbf{r}=\mathbf{a}. \tag{34}$$

Writing the inner product in this last result explicitly in terms of the geometric product,

$$\frac{1}{2}(\mathbf{a}\nabla+\nabla\mathbf{a})\mathbf{r}=\mathbf{a}, \tag{35}$$

and using associativity and (30) to evaluate the first term,

$$(\mathbf{a}\nabla)\mathbf{r}=\mathbf{a}(\nabla\mathbf{r})=3\mathbf{a}, \tag{36}$$

we deduce the derivative

$$\nabla(\mathbf{ar})=-\mathbf{a}. \tag{37}$$

Adding Eqs. (33) and (37) and dividing by 2 gives the derivative

$$\nabla(\mathbf{a}\cdot\mathbf{r})=\mathbf{a}. \tag{38}$$

Subtracting (33) from (37), dividing by 2, and then writing separate equations for the vector and trivector parts of the resulting multivector equation, gives

$$\nabla\cdot(\mathbf{a}\wedge\mathbf{r})=2\mathbf{a} \tag{39}$$

and

$$\nabla\wedge(\mathbf{a}\wedge\mathbf{r})=0. \tag{40}$$

The wedge products may be rewritten in terms of the more familiar cross product to give

$$\nabla\times(\mathbf{a}\times\mathbf{r})=2\mathbf{a} \tag{41}$$

and

$$\nabla\cdot(\mathbf{a}\times\mathbf{r})=0. \tag{42}$$

The derivative of $r\equiv|\mathbf{r}|$ is the unit radial vector:

$$\nabla r=\hat{\mathbf{r}}. \tag{43}$$

This follows by again considering the derivative defined by (5) with the cube oriented so that one pair of faces is perpendicular to $\mathbf{r}$. These faces then contribute a unit radial vector to the sum while the others sum to zero by symmetry. This derivative may alternatively be calculated by using previous results to evaluate the derivative of $r=(\mathbf{r}\cdot\mathbf{r})^{1/2}$.

All of these results could alternatively be derived by introducing Cartesian coordinates x, y, z and a set of orthonormal basis vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$, writing the derivative operator in terms of these quantities and ordinary partial derivatives as

$$\nabla=\hat{\mathbf{x}}\frac{\partial}{\partial x}+\hat{\mathbf{y}}\frac{\partial}{\partial y}+\hat{\mathbf{z}}\frac{\partial}{\partial z}, \tag{44}$$

and identifying the basis vectors as elements of the associated geometric algebra.

VI. ELECTROMAGNETIC FIELDS AND SOURCES

Let $\mathbf{E}$ and $\mathbf{B}$ be vector fields representing the conventional electric and magnetic fields. The magnetic field is actually a "axial" or "pseudo-" vector in conventional vector analysis, so we define the associated bivector field

$$\hat{B} \equiv i\mathbf{B}, \tag{45}$$

represented by an oriented plane segment perpendicular to the vector **B**. The bivector field $\hat{B}$ has the correct symmetry under space inversion and so is a more faithful representation of the magnetic field than the vector field **B**.

We also note that the Lorentz force due to a magnetic field,

$$\mathbf{f} = q\mathbf{v} \times \mathbf{B} = -q\mathbf{v} \cdot \hat{B}, \tag{46}$$

is easily interpreted geometrically in terms of the bivector $\hat{B}$: **v** is projected onto the plane of $\hat{B}$, rotated by 90° in the plane of $\hat{B}$ in the direction described by the circulation of $\hat{B}$, and multiplied by the magnitude $-qB$. The force on the most accessible charged particle, the electron, is therefore in the direction of $\mathbf{v} \cdot \hat{B}$ so that its velocity simply tends to rotate in the plane of $\hat{B}$ in the direction given by the sense of circulation of $\hat{B}$. We do not need to use the additional abstraction of the cross product and its right hand rule.

We would like to concisely represent the physical totality of these electromagnetic fields. We will do this by simply adding them together, defining the total electromagnetic field F as[2]

$$F \equiv \mathbf{E} + \hat{B}. \tag{47}$$

Since the electric and magnetic field parts have different grades (vector and bivector), we can conveniently extract either one anytime we want.

Electromagnetic fields interact with charges, both influencing and being influenced by them. We may characterize a set of discrete charges or a smooth charge continuum with a charge density $\rho(\mathbf{r})$, a scalar function of position **r**. Almost as important are the velocities of the charges, characterized by a vector current density $\mathbf{J}(\mathbf{r})$.

We would like to represent all of the necessary information about charged particles—their charge and current densities—with one mathematical quantity. The simplest possibility is just a sum or difference of charge and current densities. Subtraction turns out to work: we define the field source S as the mixed quantity

$$S \equiv \rho - \mathbf{J}. \tag{48}$$

The negative sign is associated with the signature of spacetime in a relativistic formulation. Again, given S we can quickly identify the charge and current densities anytime we like because they are of different grade.

VII. MAXWELL'S EQUATION

The simplest differential equation we can write relating a field F to a source S through space and time derivatives of the field is

$$(\nabla + \partial_t)F = S, \tag{49}$$

where $\partial_t F \equiv \partial F/\partial t$. This is in fact Maxwell's equation (time is measured in units of length such that $c=1$ in order to avoid factors of c). We will show that this equation is equivalent to the conventional expression using vector algebra.

Before continuing with our nonrelativistic formulation of electrodynamics we make some remarks about the relativistic formulation. In a relativistic formulation of electrodynamics expressed using the geometric calculus of four-dimensional spacetime, the total electromagnetic field F is a single bivector field. Any particular observer has a four-vector velocity u that defines a corresponding perpendicular three-dimensional spacelike hyperplane. Given any particular four-velocity u, the electromagnetic bivector field F can be written as a sum of two terms, a bivector that contains u as a factor and a bivector that does not contain u. These two terms are the electric and magnetic fields as seen by an observer with four-velocity u. The source S is a vector in four-dimensional spacetime whether using conventional vector algebra or geometric algebra. The geometric product of S with the observer's four-velocity u gives the scalar charge density ρ and the spacetime bivector current density J (which appears to us, as we move through spacetime, as the three-vector **J**). Maxwell's equation expressed with this spacetime geometric calculus is

$$\Box F = S, \tag{50}$$

where $\Box$ is the geometric derivative in spacetime. Multiplying (50) by the four-velocity of any particular observer gives (49), with the electric and magnetic parts of the electromagnetic field correctly separated as seen by that observer.

Although this is a powerful approach, the details are easier to understand after gaining some familiarity with a description using three-dimensional space as presented in this paper. The reader is encouraged to read the relativistic formulation detailed in Ref. 2. For the rest of this paper we will consider only the nonrelativistic formulation of electromagnetism using the geometric algebra of three-dimensional space and treating time as an independent parameter.

To show the equivalence between (49) and the conventional expression of Maxwell's equations using vector calculus, we begin by explicitly writing F in terms of **E** and **B** and expanding the geometric product involving the differential operator in terms of the inner and outer products:

$$\begin{aligned}(\nabla + \partial_t)F &= (\nabla + \partial_t)(\mathbf{E} + i\mathbf{B}) \\ &= \nabla \cdot \mathbf{E} + \nabla \wedge \mathbf{E} + \partial_t \mathbf{E} + \nabla \cdot (i\mathbf{B}) + \nabla \wedge (i\mathbf{B}) \\ &\quad + \partial_t i\mathbf{B}.\end{aligned} \tag{51}$$

Some identities needed here and easily proved using corresponding identities from geometric algebra and the definition of the geometric derivative are

$$\nabla \cdot (i\mathbf{B}) = i(\nabla \wedge \mathbf{B}) \tag{52}$$

and

$$\nabla \wedge (i\mathbf{B}) = i(\nabla \cdot \mathbf{B}), \tag{53}$$

true for any vector **B**. Applying these to (51) we have

$$\nabla \cdot \mathbf{E} + \nabla \wedge \mathbf{E} + \partial_t \mathbf{E} + i(\nabla \wedge \mathbf{B}) + i(\nabla \cdot \mathbf{B}) + \partial_t i\mathbf{B} = \rho - \mathbf{J}. \tag{54}$$

Equating terms of equal grade on each side of this equation gives four equations, one each for scalar, vector, bivector, and trivector terms:

$$\nabla \cdot \mathbf{E} = \rho$$

$$+\partial_t \mathbf{E} + i(\nabla \wedge \mathbf{B}) = -\mathbf{J}$$

$$+\nabla \wedge \mathbf{E} + \partial_t i\mathbf{B} = 0$$

$$i(\nabla \cdot \mathbf{B}) = 0. \quad (55)$$

Rewriting the wedge product in terms of the cross product we have the very familiar equations

$$\nabla \cdot \mathbf{E} = \rho$$

$$+\partial_t \mathbf{E} - \nabla \times \mathbf{B} = -\mathbf{J}$$

$$+\nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0$$

$$\nabla \cdot \mathbf{B} = 0. \quad (56)$$

VIII. PLANE-WAVE SOLUTIONS TO MAXWELL'S EQUATION

If there are no sources, Maxwell's equation is

$$(\nabla + \partial_t)F = 0. \quad (57)$$

In this section, we will consider the simple trial solution,

$$F = Ae^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}, \quad (58)$$

where A is some constant multivector (possibly a sum of simple multivectors), $\mathbf{k}$ is a constant vector, and ω is a constant scalar frequency. Note that we are *not* using the usual real-part convention: we will see that all mathematical objects contained in (58) have physical significance.

Substituting (58) into (57) and taking the derivatives, we have

$$i(\mathbf{k} - \omega)Ae^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} = 0. \quad (59)$$

We simplify this by multiplying on the left with the inverse $-i$ of i and on the right by the inverse $\exp[-i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$ of $\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$, yielding

$$(\mathbf{k} - \omega)A = 0. \quad (60)$$

Since the electromagnetic field F is the sum of a vector and bivector by assumption, let us assume that the coefficient A of our trial solution is the sum of a constant vector $\mathbf{V}$ and constant bivector $\hat{W}$:

$$A = \mathbf{V} + \hat{W}. \quad (61)$$

Substituting (61) into (60), writing out the geometric products in terms of inner and outer products, and grouping scalars, vectors, bivectors, and trivectors, we have

$$0 = (\mathbf{k} - \omega)(\mathbf{V} + \hat{W})$$

$$= (\mathbf{k}\cdot\mathbf{V}) + (-\omega\mathbf{V} + \mathbf{k}\cdot\hat{W}) + (\mathbf{k}\wedge\mathbf{V} - \omega\hat{W})$$

$$+ (\mathbf{k}\wedge\hat{W}). \quad (62)$$

This is equivalent to four separate equations, one for each grade. The scalar equation

$$0 = \mathbf{k}\cdot\mathbf{V} \quad (63)$$

says the vectors $\mathbf{k}$ and $\mathbf{V}$ are perpendicular. The trivector equation

$$0 = \mathbf{k}\wedge\hat{W} \quad (64)$$

says $\mathbf{k}$ is tangential to the bivector $\hat{W}$. The bivector equation

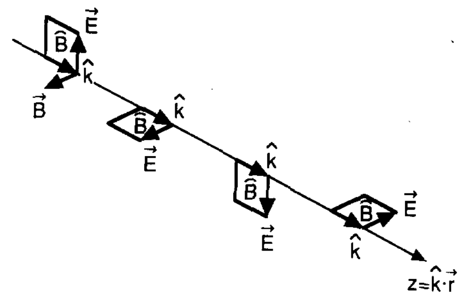

Fig. 2. A graphic representation of a circularly polarized electromagnetic wave, showing the electric vector field $\mathbf{E}$ and magnetic bivector field $\hat{B} = i\mathbf{B}$ as functions of position $z \equiv \hat{k}\cdot\mathbf{r}$ in the propagation direction $\hat{k}$ at an instant of time.

$$0 = \mathbf{k}\wedge\mathbf{V} - \omega\hat{W} \quad (65)$$

says that $\hat{W}$ is the bivector $(\mathbf{k}\wedge\mathbf{V})/\omega$. The vector equation

$$0 = -\omega\mathbf{V} + \mathbf{k}\cdot\hat{W} \quad (66)$$

says that $\mathbf{V}$ is the vector $(\mathbf{k}\cdot\hat{W})/\omega$, also tangential to $\hat{W}$: its direction is obtained by projecting $\mathbf{k}$ onto the plane of $\hat{W}$ and rotating the result by 90°. But we have previously concluded from (64) that $\mathbf{k}$ is also in the plane of $\hat{W}$, so projecting is not necessary; $\mathbf{k}$ and $\mathbf{V}$ are therefore both in the plane of $\hat{W}$ and are perpendicular.

We can substitute (66) into (65) and easily evaluate the first term: Since $\mathbf{k}$ and $\mathbf{V}$ are in the plane of $\hat{W}$, $\mathbf{k}\wedge\mathbf{V}$ is a multiple of $\hat{W}$, and since they are perpendicular, the magnitude of $\mathbf{k}\wedge\mathbf{V}$ is simply equal to a product of the magnitudes of the factors. Then we have

$$0 = (k^2/\omega)\hat{W} - \omega\hat{W}, \quad (67)$$

which implies

$$|\mathbf{k}| = \omega. \quad (68)$$

This says the phase velocity is 1. (Recall that the units of time have been chosen so that the velocity of light is unity, $c = 1$.)

We conclude that (1) the wave amplitude A must be of the form

$$A = \mathbf{V} + \hat{W} = \mathbf{V} + \hat{\mathbf{k}}\wedge\mathbf{V}, \quad (69)$$

where $\mathbf{V}$ is some vector perpendicular to $\mathbf{k}$, and (2) $\omega = |\mathbf{k}|$.

With the form of A now known, we can return to interpret the trial expression (57) for the electromagnetic field. Using Euler's expression for the exponential in terms of trigonometric functions and collecting together vector and bivector terms, we have

$$F = \mathbf{E} + i\mathbf{B}$$

$$= (\mathbf{V} + \hat{W})[\cos(\mathbf{k}\cdot\mathbf{r} - \omega t) + i\sin(\mathbf{k}\cdot\mathbf{r} - \omega t)]$$

$$= [\mathbf{V}\cos(\mathbf{k}\cdot\mathbf{r} - \omega t) + (\hat{W}i)\sin(\mathbf{k}\cdot\mathbf{r} - \omega t)]$$

$$+ [\hat{W}\cos(\mathbf{k}\cdot\mathbf{r} - \omega t) + (\mathbf{V}i)\sin(\mathbf{k}\cdot\mathbf{r} - \omega t)]. \quad (70)$$

This describes a circularly polarized wave. The first pair of terms describes an electric vector field rotating from $\mathbf{V}$ to the perpendicular vector $\hat{W}i$ to $-\mathbf{V}$ and so on. The second pair of terms describes a magnetic bivector field rotating from $\hat{W}$ to the perpendicular bivector $\mathbf{V}i$ to $-\hat{W}$ and so on. See Fig. 2.

Note that every mathematical object symbolized by the expression (70) for the field F corresponds to a physical, geometrically visualizable quantity; we are not "taking the real part" as must be done when interpreting complex vectors as conventionally used in vector analysis.

A linearly polarized wave is simply the sum of a right and left circularly polarized wave:

$$F=\frac{A}{2}\,(e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}+e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}). \tag{71}$$

Expanding as in (70), the sin terms cancel. In the resulting expression we may write the bivector $\hat{W}$ either as a wedge product or in terms of the cross product:

$$\begin{aligned}F&=\mathbf{E}+i\mathbf{B}\\&=\mathbf{V}\cos(\mathbf{k}\cdot\mathbf{r}-\omega t)+\hat{W}\cos(\mathbf{k}\cdot\mathbf{r}-\omega t)\\&=\mathbf{V}\cos(\mathbf{k}\cdot\mathbf{r}-\omega t)+(\hat{k}\wedge\mathbf{V})\cos(\mathbf{k}\cdot\mathbf{r}-\omega t)\\&=\mathbf{V}\cos(\mathbf{k}\cdot\mathbf{r}-\omega t)+i(\hat{k}\times\mathbf{V})\cos(\mathbf{k}\cdot\mathbf{r}-\omega t).\end{aligned} \tag{72}$$

This explicitly shows that the vector electric field and the magnetic bivector field are in phase and do not change directions.

Comparison of the first and last lines of (72) shows that the magnetic field vector $\mathbf{B}$ is perpendicular to the unit wave vector $\hat{\mathbf{k}}$ and to the electric field $\mathbf{E}$; although visualizing the magnetic field as a bivector field $\hat{B}$ tangential to $\hat{\mathbf{k}}$ and $\mathbf{E}$ corresponds more closely with the physical significance of the magnetic field as a source of the Lorentz force, discussed at the beginning of Sec. VII.

We emphasize that although expressions like (58) and (71) appear identical to expressions using complex vector notation, they have quite a different interpretation as we are not discarding an imaginary part and we need no additional expressions to give us the magnetic field.

IX. THE GENERAL SOLUTION TO MAXWELL'S EQUATION WITH SOURCES

We now find a few expressions for the solution to Maxwell's equation for the field F due to any source S.

First we find the Green's function G for the Maxwell equation, satisfying

$$(\nabla+\partial_t)G(\mathbf{r},t)=4\pi\delta^3(\mathbf{r})\delta(t). \tag{73}$$

This is easy: The Green's functions $g^{\pm}$ of the familiar scalar wave equation

$$(\nabla^2-\partial_t^2)g(\mathbf{r},t)=4\pi\delta^3(\mathbf{r})\delta(t) \tag{74}$$

are

$$g^{\pm}(\mathbf{r},t)=-\frac{\delta(t\pm r)}{r}, \tag{75}$$

though for most physical applications only g^- is relevant. Let us call this Green's function g. By factoring the differential operator in (74) to get

$$(\nabla+\partial_t)(\nabla-\partial_t)g(\mathbf{r},t)=4\pi\delta^3(\mathbf{r})\delta(t), \tag{76}$$

and comparing with Eq. (73) for G, it is immediately apparent that

$$G(\mathbf{r},t)=(\nabla-\partial_t)g(\mathbf{r},t)=-(\nabla-\partial_t)\,\frac{\delta(t-r)}{r}. \tag{77}$$

Using this result we may write an expression for the electromagnetic field F in terms of the source S:

$$\begin{aligned}F(\mathbf{r},t)&=\int d^3r'\int dt'G(|\mathbf{r}-\mathbf{r}'|,t-t')S(\mathbf{r}',t')\\&=-\int d^3r'\int dt'\left((\nabla_\eta-\partial_\tau)\right.\\&\quad\left.\times\frac{\delta(\tau-\eta)}{\eta}\right)_{\substack{\tau=t-t'\\ \eta=\mathbf{r}-\mathbf{r}'}}S(\mathbf{r}',t')\end{aligned} \tag{78}$$

After rewriting the derivatives with respect to $\eta=\mathbf{r}-\mathbf{r}'$ and $\tau=t-t'$ in terms of derivatives with respect to $\mathbf{r}$ and t, we may do the integral over t'. This yields

$$F(\mathbf{r},t)=-(\nabla_{\mathbf{r}}-\partial_t)\int d^3r'\,\frac{S(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}. \tag{79}$$

This last result is equivalent to the usual expression of the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ in terms of scalar and vector potentials. To show this we define

$$\phi\equiv\int d^3r'\,\frac{\rho(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}\quad\text{and}$$

$$\mathbf{A}\equiv\int d^3r'\,\frac{\mathbf{J}(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|} \tag{80}$$

and explicitly write F in terms of $\mathbf{E}$ and $\mathbf{B}$. Then Eq. (79) may be written

$$\mathbf{E}(\mathbf{r},t)+i\mathbf{B}(\mathbf{r},t)=-(\nabla_r-\partial_t)(\phi-\mathbf{A}). \tag{81}$$

Expanding the geometric derivatives as inner and outer derivatives and grouping terms of equal grade, this is

$$\begin{aligned}\mathbf{E}(\mathbf{r},t)+i\mathbf{B}(\mathbf{r},t)=&+\{\nabla_r\cdot\mathbf{A}+\partial_t\phi\}+\{-\nabla_r\phi-\partial_t\mathbf{A}\}\\&+i\{\nabla_r\times\mathbf{A}\}.\end{aligned} \tag{82}$$

The scalar, vector, and bivector parts of this equation are the same as the Lorentz condition and expressions for the electric and magnetic fields in conventional vector calculus.

Another expression of the electromagnetic fields may be obtained by first explicitly evaluating the derivatives in the Green's function (77). This gives

$$G(\mathbf{r},t)=\frac{\hat{\mathbf{r}}}{r^2}\delta(t-r)+\frac{\hat{\mathbf{r}}+1}{r}\delta'(t-r), \tag{83}$$

where $\delta'(x)\equiv(d/dx)\delta(x)$. In terms of this expression, the solution F to Maxwell's equation is

$$\begin{aligned}F(\mathbf{r},t)&=\int d^3r'\int dt'G(|\mathbf{r}-\mathbf{r}'|,t-t')S(\mathbf{r}',t')\\&=\int d^3r'\int dt'\left(\frac{\hat{\eta}}{\eta^2}\delta(\tau-\eta)\right.\\&\quad\left.+\frac{\hat{\eta}+1}{\eta}\delta'(\tau-\eta)\right)_{\substack{\tau=t-t'\\ \eta=\mathbf{r}=\mathbf{r}'}}S(\mathbf{r}',t').\end{aligned} \tag{84}$$

We may immediately integrate over the variable t', integrating by parts on the second term after using

$$\delta'(\tau-\eta)=\frac{\partial}{\partial\tau}\delta(\tau-\eta)=-\frac{\partial}{\partial t'}\delta(\tau-\eta),\quad \tau=t-t'. \tag{85}$$

For notational convenience we will indicate an ordinary partial derivative with respect to time by a subscript on the function rather than with a prefix of ∂ that bears the subscript; e.g., $S_{t'}\equiv\partial_{t'}S$. Then we have

$$F(\mathbf{r},t)=\int d^3r'\left(\frac{\hat{\boldsymbol{\eta}}}{\eta^2}S(\mathbf{r}',t')+\frac{\hat{\boldsymbol{\eta}}+1}{\eta}S_{t'}(\mathbf{r}',t')\right)_{\substack{t'=t-\eta\\ \boldsymbol{\eta}=\mathbf{r}-\mathbf{r}'}}. \tag{86}$$

This is equivalent to Jefimenko's expressions for the electric and magnetic fields, as recently discussed in these pages.[6] We may show this by explicitly writing the source S in (86) in terms of the charge and current densities ρ and $\mathbf{J}$ and writing out all geometric products as sums of inner and outer products. Doing this and grouping resulting terms by grade (scalar, vector, and bivector), (86) becomes

$$\begin{aligned}F(\mathbf{r},t)=\int d^3r'\Bigg[&\left(-\frac{\hat{\boldsymbol{\eta}}\cdot\mathbf{J}}{\eta^2}-\frac{\hat{\boldsymbol{\eta}}\cdot\mathbf{J}_{t'}}{\eta}+\frac{\rho_{t'}}{\eta}\right)\\ &+\left(\frac{\hat{\boldsymbol{\eta}}}{\eta^2}\rho+\frac{\hat{\boldsymbol{\eta}}}{\eta}\rho_{t'}-\frac{\mathbf{J}_{t'}}{\eta}\right)\\ &-\left(\frac{\hat{\boldsymbol{\eta}}\wedge\mathbf{J}}{\eta^2}+\frac{\hat{\boldsymbol{\eta}}\wedge\mathbf{J}_{t'}}{\eta}\right)\Bigg]_{\substack{t'=t-\eta\\ \boldsymbol{\eta}=\mathbf{r}-\mathbf{r}'}}.\end{aligned} \tag{87}$$

Writing $\mathbf{E}+i\mathbf{B}$ for F and writing the outer products in terms of the cross product, this is

$$\begin{aligned}\mathbf{E}+i\mathbf{B}=\int d^3r'\Bigg[&\left(-\frac{\hat{\boldsymbol{\eta}}\cdot\mathbf{J}}{\eta^2}-\frac{\hat{\boldsymbol{\eta}}\cdot\mathbf{J}_{t'}}{\eta}+\frac{\rho_{t'}}{\eta}\right)\\ &+\left(\frac{\hat{\boldsymbol{\eta}}}{\eta^2}\rho+\frac{\hat{\boldsymbol{\eta}}}{\eta}\rho_{t'}-\frac{\mathbf{J}_{t'}}{\eta}\right)\\ &-i\left(\frac{\hat{\boldsymbol{\eta}}\times\mathbf{J}}{\eta^2}+\frac{\hat{\boldsymbol{\eta}}\times\mathbf{J}_{t'}}{\eta}\right)\Bigg]_{\substack{t'=t-\eta\\ \boldsymbol{\eta}=\mathbf{r}-\mathbf{r}'}}.\end{aligned} \tag{88}$$

The scalar part of this equation (zero on the left and the first parenthesis on the right) is an expression of the continuity condition, while the vector and bivector parts of this equation are identical to Jefimenko's expressions for the electric and magnetic fields in terms of the charge and current sources.

X. ENERGY-MOMENTUM

We first recall the mathematical operation of *reversion* or *conjugation* as follows. A general member A of the geometric algebra of space may be a sum of scalar, vector, bivector, and trivector terms. The bivector and trivector terms may be written as outer products between vectors. We define the reverse or conjugate $A^\dagger$ of A to be the quantity obtained by reversing the order of vector factors in every such outer product in A. The symmetry of the outer product implies that bivector and trivector terms change sign under conjugation while all other terms are unchanged.

With this definition, we evaluate the product of the conjugate $F^\dagger$ of the electromagnetic field with the field F:

$$\begin{aligned}\frac{1}{2}F^\dagger F&\equiv\frac{1}{2}(\mathbf{E}-i\mathbf{B})(\mathbf{E}+i\mathbf{B})\\ &=\frac{1}{2}[\mathbf{EE}-i\mathbf{B}i\mathbf{B}+(\mathbf{E}i\mathbf{B}-i\mathbf{BE})]\\ &=\frac{1}{2}[\mathbf{EE}-ii\mathbf{BB}+i(\mathbf{EB}-\mathbf{BE})]\\ &=\frac{1}{2}(|\mathbf{E}|^2+|\mathbf{B}|^2)+i(\mathbf{E}\wedge\mathbf{B})\\ &=\frac{1}{2}(|\mathbf{E}|^2+|\mathbf{B}|^2)-(\mathbf{E}\times\mathbf{B}).\end{aligned} \tag{89}$$

We see that the scalar and vector parts of this are equal to the energy and (negative) momentum densities. We will call $(1/2)F^\dagger F$ simply the energy-momentum density. The integral of the energy-momentum over all space will be called the total energy-momentum.

As an example we will evaluate the energy-momentum of the plane wave of the previous section. Using the fact that $\mathbf{V}$ is known to be perpendicular to $\mathbf{k}$ and so anticommutes, and that $\mathbf{k}$ commutes with all scalars and pseudoscalars (including $\mathbf{k}\cdot\mathbf{r}$) and therefore with the exponential of a pseudoscalar [such as $\exp(i\mathbf{k}\cdot\mathbf{r})$], we have

$$\begin{aligned}\frac{1}{2}F^\dagger F&=\frac{1}{2}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}(\mathbf{V}+\mathbf{V}\hat{\mathbf{k}})(\mathbf{V}+\hat{\mathbf{k}}\mathbf{V})e^{+i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\\ &=\frac{1}{2}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}(\mathbf{VV}+\mathbf{V}\hat{\mathbf{k}}\hat{\mathbf{k}}\mathbf{V}+2\mathbf{V}\hat{\mathbf{k}}\mathbf{V})e^{+i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\\ &=\frac{1}{2}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}(|\mathbf{V}|^2+\mathbf{V}|\hat{\mathbf{k}}|^2\mathbf{V}\\ &\quad-2|\mathbf{V}|^2\hat{\mathbf{k}})e^{+i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\\ &=e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}(|\mathbf{V}|^2-|\mathbf{V}|^2\hat{\mathbf{k}})e^{+i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\\ &=|\mathbf{V}|^2-|\mathbf{V}|^2\hat{\mathbf{k}}.\end{aligned} \tag{90}$$

This states the well-known result for a circularly polarized plane wave: The energy density is just V^2, the squared peak magnitude of the electric or magnetic field, and the momentum density is V^2 times a unit vector in the propagation direction (recall that time is measured so that $c=1$).

We will now evaluate the time derivative of the total energy-momentum. We can do this after the following considerations of Maxwell's equation. Maxwell's equation times $F^\dagger$ on the left is

$$F^\dagger(\dot{\nabla}+\dot{\partial}_t)\dot{F}=F^\dagger S, \tag{91}$$

where the overdots indicate that both derivatives are acting in the conventional direction on the F to the right. The *conjugate* of Maxwell's equation, times F on the right, is

$$\dot{F}^\dagger(\dot{\nabla}+\dot{\partial}_t)F=S^\dagger F, \tag{92}$$

where the overdots now indicate that the derivatives are acting on the $F^\dagger$ to the left. Adding these two equations we have

$$(\dot{F}^\dagger\dot{\nabla}F+F^\dagger\dot{\nabla}\dot{F})+(\dot{F}^\dagger\dot{\partial}_tF+F^\dagger\dot{\partial}_t\dot{F})=S^\dagger F+F^\dagger S. \tag{93}$$

Using the product rule (21) the pairs of derivatives may be combined to give

$$\dot{F}^\dagger\dot{\nabla}\dot{F}+\dot{F}^\dagger\dot{\partial}\dot{F}=S^\dagger F+F^\dagger S, \tag{94}$$

where the overdots indicate that the derivatives operate both to the left and right.

Now we integrate this equation over some volume τ. The first term is a total derivative and so its volume integral equals a surface integral. For similarity in appearance we have treated the time derivative terms like the space derivatives with respect to the order of factors, but since the time derivative ∂_t is a scalar its order does not actually matter. Then after integrating and dividing by 2 we have

$$\partial_t\left(\frac{1}{2}\int_\tau d\bar{\tau}F^\dagger F\right)=\int_\tau d\bar{\tau}\left(\frac{S^\dagger F+F^\dagger S}{2}\right)-\frac{1}{2}\int_\sigma F^\dagger d\hat{\sigma}F. \tag{95}$$

We can rewrite the first integrand on the right in terms of ρ, $\mathbf{J}$, and $\mathbf{E}$:

$$\begin{aligned}\frac{S^\dagger F+F^\dagger S}{2}&=\frac{1}{2}\left[(\rho-\mathbf{J})(\mathbf{E}+i\mathbf{B})+(\mathbf{E}-i\mathbf{B})(\rho-\mathbf{J})\right]\\&=-\mathbf{E}\cdot\mathbf{J}+\rho\mathbf{E}-\mathbf{J}\cdot(i\mathbf{B})\\&=-\mathbf{E}\cdot\mathbf{J}+\rho\mathbf{E}-i(\mathbf{J}\wedge\mathbf{B}).\end{aligned} \tag{96}$$

The first term is the only scalar and the next two are the vector terms; the bivector and trivector terms each sum to zero. Using this result and assuming the surface integral is zero, we may write (95) as

$$\begin{aligned}&\partial_t\left[\frac{1}{2}\int d^3r(|\mathbf{E}|^2+|\mathbf{B}|^2)\right]+\partial_t\left[\int d^3ri(\mathbf{E}\wedge\mathbf{B})\right]\\&=-\int d^3r(\mathbf{E}\cdot\mathbf{J})+\int d^3r(\rho\mathbf{E}-i(\mathbf{J}\wedge\mathbf{B})).\end{aligned} \tag{97}$$

The scalar part of this equation is

$$\partial_t\left[\frac{1}{2}\int d^3r(|\mathbf{E}|^2+|\mathbf{B}|^2)\right]=-\int d^3r(\mathbf{E}\cdot\mathbf{J}). \tag{98}$$

This says that the rate of change of energy of the electromagnetic field equals minus the work done on the charge distribution. Combined with the work-energy theorem of mechanics, this implies conservation of total energy. The vector part of Eq. (97), written in terms of the more familiar cross product, is

$$\partial_t\left[\int d^3r(\mathbf{E}\times\mathbf{B})\right]=-\int d^3r(\rho\mathbf{E}+\mathbf{J}\times\mathbf{B}). \tag{99}$$

This says that the rate of change of the momentum of the electromagnetic field equals the negative of the Lorentz force exerted by the field on the charge distribution. Combined with Newton's third law, this implies conservation of total momentum.

We can show that the surface integral in (95) describes rates of change of energy and momentum corresponding to energy and momentum flowing into or out of the volume being considered, but we will not pursue this here.

XI. CONCLUDING REMARKS

The simplest possible differential equation involving the geometric derivative and a time derivative turns out to be equivalent to all four of Maxwell's equations. Plane-wave solutions in the source-free case are formally identical to the simple solutions of the corresponding one-dimensional problem using complex numbers, but contain all the details of the full 3D vector solution. The general solution with sources is easily constructed from the solution to the scalar wave equation in 3D space. This is all done with an algebraic system and associated calculus that are fairly straightforward extensions, or one might argue completions, of conventional vector algebra and calculus.

No new physics comes of this description of electrodynamics. But geometric algebra has advantages over conventional vector algebra that are very similar to the advantages of conventional vector algebra over equations describing separate Cartesian components. In both of these cases the conceptual complexity and the difficulty and length of algebraic manipulations are reduced by using the more compact, concise formulation.

[a]e-mail: tvold1@cc.swarthmore.edu

[1]Terje G. Vold, "An introduction to geometric algebra with an application in rigid body mechanics," Am. J. Phys **61**, 491–504 (1993).

[2]David Hestenes, *Spacetime Algebra* (Gordon and Breach, New York, 1966).

[3]David Hestenes and Garret Sobczyk, *Clifford Algebra to Geometric Calculus* (Reidel, Boston, 1984).

[4]Bernard Jancewicz, *Multivectors and Clifford Algebra in Electrodynamics* (World Scientific, Teanack, New Jersey, 1988).

[5]David Hestenes, *New Foundations for Classical Mechanics* (Reidel, Boston, 1986).

[6]David J. Griffiths and Mark A. Heald, "Time-dependent generalizations of the Biot-Savart and Coulomb Laws," Am. J. Phys. **59** (2), 111–7 (1991).

NEWTON'S DESIGN

My Design in this Book is not to explain the Properties of Light by Hypotheses, but to propose and prove them by Reason and Experiments.

Isaac Newton, Opticks (1730), 4th ed. (Reprinted by Dover, New York, 1952), p. 1.