An introduction to geometric algebra with an application in rigid body mechanics

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(Received 8 September 1992; accepted 16 October 1992)

This paper presents a tutorial of geometric algebra, a very useful but generally unappreciated extension of vector algebra. The emphasis is on physical interpretation of the algebra and motives for developing this extension, and not on mathematical rigor. The description of rotations is developed and compared with descriptions using vector and matrix algebra. The use of geometric algebra in physics is illustrated by solving an elementary problem in classical mechanics, the motion of a freely spinning axially symmetric rigid body.

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I. INTRODUCTION

Around 1880 Willard J. Gibbs invented[1] his vector algebra, building upon work by Grassmann and Hamilton. Until then physical ideas dealing with geometrical or spatial relationships—such as the theory of sound[2] and Maxwell's new electrodynamics[3]—were generally expressed by writing one equation for each component of a Cartesian coordinate system, or by using a more complex or restrictive method such as velocity potentials for sound and quaternions for electromagnetism. Gibbs' vector algebra and its associated calculus were very successful and caught on like wildfire.

Vector algebra is now taken for granted by most physicists, almost as if it were part of nature rather than a recent invention to more conveniently *describe* nature. It may seem reasonable to conclude that after a century of very successful use, Gibbs' vector algebra has proven itself to be the optimal mathematical system for algebraic expression of geometrical information.

Other systems have, however, also been developed: in particular, Cartan's exterior algebra of forms[4] and Hestenes' geometric algebra.[5–7] The algebra of forms, developed by Cartan in the 1920s,[8] has been extensively used by physicists in general relativity since the 1970s. Geometric algebra, developed by Hestenes in the 1960s, is being used by an growing number of physicists today[9] but is still not widely known. It is easy to learn and use, includes both Gibbs' vector algebra and Cartan's algebra of forms as natural subsystems, and has significant advantages over both.

This essay provides an introduction to the geometric algebra of three-dimensional space. It is applied to the mechanics of a freely rotating rigid body as an example of its use and advantage over conventional vector algebra. A following paper[10] describes its associated calculus and illustrates its use by applying it to electrodynamics.

II. IDENTIFICATION OF THE TASK

In this paper geometric algebra will be developed from the point of view of trying to construct an algebra that will be useful and easy to use in geometrical problems. (See Ref. 6 for a somewhat different but much more extensive treatment at about the same level.) This will naturally lead us to the basic concepts of conventional vector algebra plus a few more. The resulting algebra integrates the ideas of axial vectors and pseudoscalars with vectors and scalars at its foundations, and has a single fundamental multiplicative product. It contains conventional vector algebra as a subsystem and so may be viewed as an extension, or perhaps more appropriately a completion, of vector algebra.

An "algebra" is a mathematical system using symbols that are strung together in lines to make equations and manipulated according to given rules. Quantitatively precise relationships are expressed. By "geometrical" we mean involving spatial qualities associated with physical objects such as relative position, length, area, and volume. We generally represent geometrical information first by drawing a picture. This usually contains a great deal of qualitative or semiquantitative information, but little or no

quantitatively precise information. We want to precisely represent geometrical information with an algebra that we will invent for this purpose.

We will do this by first identifying some geometrical and physical quantities that we might like to represent with an algebra and then defining operations of addition and multiplication between these quantities. The general idea will be to try to incorporate as many simple, common-sense ideas about geometry as possible in the basic definitions of elements and operations, while simultaneously keeping the algebra as simple as possible and similar to the familiar algebra of real numbers. The resulting algebra is formally a Clifford algebra; "geometric algebra" refers to the combination of this algebraic structure and the geometrical interpretation described here.

The emphasis in this paper is on a physicist's motivation for construction of this system and on the correspondence between the algebra and its geometrical interpretation, and not on mathematical rigor or conciseness. Details along with a formal axiomatic development of the mathematical system are available elsewhere.[7]

III. THE MATHEMATICAL ELEMENTS OF THE ALGEBRA AND THE PHYSICAL QUANTITIES THEY MAY REPRESENT

First we present a qualitative overview of the types of physical quantities that we would like to be able to represent in the algebra. Each type will be represented by a different type of mathematical object. In this section we will only *describe* these mathematical objects by giving desired correspondences with physical or geometrical quantities. They will be mathematically *defined* only implicitly by the algebraic rules they obey, discussed in the following sections. But we will choose these rules so that these objects and their rules correspond in simple common-sense ways to various physical quantities and physical or geometrical relationships between them. We will then be able to augment the powerful geometric intuition of the human mind with the precision of an algebraic system.

We will represent four types of quantities in this algebra, as follows.

(1) Physical quantities with magnitude but no spatial extent, such as number of objects, mass, electric charge, etc. These will be represented by real numbers as usual. We will often call these real numbers *scalars* to emphasize that they have magnitude or scale (and sign) but no direction.

(2) Linelike physical quantities that have directions along with magnitudes, such as relative position, displacement, and velocity. These will be represented by vectors **a**, **b**, **c**, ... as usual. The archetype to keep in mind is a displacement, or equivalently, the relative position after a displacement compared with before. The standard graphic representation to keep in mind is a directed line segment—that is, an arrow.

(3) Planelike physical quantities having orientation along with magnitude, such as surface area and the act of rotation in a plane. We will call these mathematical objects *bivectors*. The archetype to keep in mind is a finite rotation, or equivalently, the relative orientation or attitude of a body after a rotation compared with before. The standard graphic representation to keep in mind is an oriented parallelogram defined by two vectors **a** and **b** with the head of **a** attached to the tail of **b** forming the bivector $\mathbf{a}\wedge\mathbf{b}$. The plane of the parallelogram is parallel to the plane of the circular path of any point on the rotating body, and its area is proportional to the magnitude of the rotation.

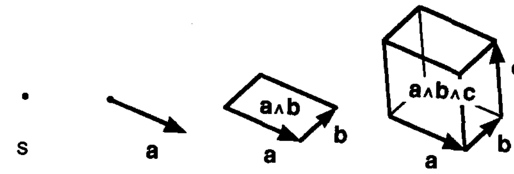

Fig. 1. Graphic representations of a scalar s, vector **a**, bivector $\mathbf{a}\wedge\mathbf{b}$, and trivector $\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}$.

(4) Last, we want to represent volumelike quantities having a choice of handedness along with magnitude. One physical example is a surface area times a velocity giving a rate of transport through the surface. The standard graphic representation to keep in mind is a parallelpiped volume defined by three vectors **a**, **b**, and **c**. The handedness of the three vectors will determine the sign. We will call these mathematical objects *trivectors*. A trivector defined by **a**, **b**, and **c** is denoted $\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}$.

Collectively we'll call these objects *multivectors* with grades 0, 1, 2, and 3. See Fig. 1.

We would like to our mathematical objects to be as simple as possible even though they may represent physical or geometrical quantities having much more detail and information. For example, a displacement of a physical object is represented by a vector that depends only on the direction of a direct path on a straight line from the initial to final location and on the total magnitude of the final displacement; it is independent of all details of the path of the physical displacement. A vector represents only the essential abstractions of direction and magnitude. All other information about the particular path of the displacement is ignored. Often a single vector can be used to represent all the information we want to abstract from a single curved physical line—its end-to-end length and magnitude—but if details of a curved physical line are important it is approximated with many small straight vectors.

Similarly, a rotation of a physical object is represented by a bivector that depends only on the orientation of the plane of a direct rotation from the initial to final attitude and on the total magnitude of the direct rotation; it is independent of all details of the path of the actual rotation. A bivector represents only the abstractions of planar orientation and magnitude. The shape of the plane segment will be ignored, although the sign of the direction of the rotation must be retained. A bivector may be graphically represented by a plane segment with one or more arrows on the perimeter indicating the sense of rotation. Any shape is appropriate, parallelogram, circle, or other, as long as it has the correct area and orientation. If we wish to represent a physical surface area with a bivector, often a single bivector can be used to represent all the information we want to abstract—its maximum projected area and corresponding orientation—but if details of a curved physical surface are important it is approximated with many small bivectors.

Of course for every plane segment we can uniquely define a vector that is perpendicular to the the plane segment and with equal magnitude. Vector algebra uses this correspondence to represent areas and area-like quantities with

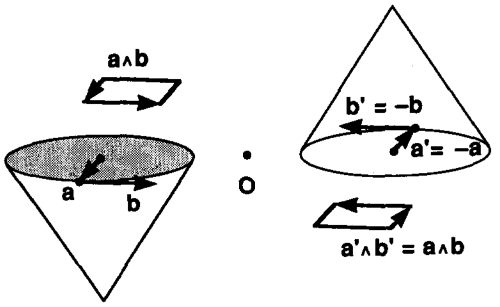

Fig. 2. A rotating object, exemplified by a cone, and its spatial inversion through an arbitrary point O. The relative position and velocity of any particle of the object, represented by the vectors $\mathbf{a}$ and $\mathbf{b}$, change sign upon inversion, becoming $\mathbf{a}'=-\mathbf{a}$ and $\mathbf{b}'=-\mathbf{b}$. Angular momentum, represented by the bivector $\mathbf{a}\wedge\mathbf{b}$, does not change sign.

vectors. In fact, most physicists are so used to this correspondence, especially for rotations, that the vector representation may seem more natural and fundamental. But this has two serious problems.

First, linelike and arealike physical quantities change differently under space inversion (reflection through a single chosen point): directed linelike quantities such as displacements and velocities are equal to the negative of the corresponding noninverted quantities, while the signs of oriented arealike quantities such as rotations and angular velocities are unchanged under inversion. See Fig. 2. This is correctly represented in geometric algebra while vector algebra handles it with the ad-hoc introduction of "axial" or "pseudo-" vectors, simply decreed to be vectors that do not change sign upon spatial inversion of the corresponding physical system.

The second, perhaps more serious problem is that this correspondence only works in three dimensions. If we try to do mechanics in two dimensions with vector algebra, we must in some ad hoc way allow vectors perpendicular to the plane—outside of our presumed 2D world—to represent quantities like angular velocity. And if we try to do mechanics in four dimensional spacetime, we find that rotations cannot be uniquely identified by an axis but require a two-dimensional hyperplane, typically specified with a 4×4 matrix. Both of these are neatly described by bivectors in geometric algebra.

With these notions of the nature of our mathematical objects in mind, we assume that any multivector A may be written as a product of its magnitude $|A|$, a real number, times a unit multivector. The unit multivector may be a scalar, vector, bivector, or a trivector in the geometric algebra associated with three-dimensional space. The only unit scalar is the number 1. There are an infinite number of unit vectors corresponding to the infinity of different directions possible, although any three mutually orthogonal unit vectors may be used as a basis just as in vector algebra. There are also an infinite number of possible unit bivectors, corresponding to the infinity of possible orientations of a surface, and again any three mutually orthogonal unit bivectors may be used as a basis for bivectors. There is only one unit volume element, equal to the outer product of any three orthonormal vectors as discussed below. The only possible "direction" for such a volume that is independent of the volume's shape is the handedness of the defining triplet of vectors, and this will correspond to the sign of the trivector.

In comparison, if we use conventional Gibbs' vector algebra we start out with only scalars and vectors. Typically we then define linear functions of vector variables—that is, tensors—and their matrix representations, and also develop vector calculus. Only much later, possibly prompted by nature's violation of parity symmetry as first described by Lee and Yang[11] and observed by Garwin[12] and Wu,[13] do we make a distinction between vectors and pseudovectors (or axial vectors) and between scalars and pseudoscalars (necessary, for example, to write down the effective Hamiltonian for the weak interaction). But geometric algebra has bivectors and trivectors built into its foundations, and these correspond directly to pseudo-vectors and pseudo-scalars.

IV. OPERATIONS BETWEEN ELEMENTS OF THE ALGEBRA AND CORRESPONDING PHYSICAL INTERPRETATIONS

Now we need to identify some physically useful operations between these physical or geometrical objects, and define corresponding operations between the multivectors. By "operation" we mean a rule that gives a third element of the algebra as an output for any two given elements as inputs. The algebra of real numbers has two operations, addition and multiplication, yielding sums and products. We will find that we can construct an algebra of multivectors that also has one kind of sum and one fundamental kind of product. We'll start will scalars as inputs and then generalize to multivectors of higher grade.

A. Operations involving scalars

We want to be able to use the scalars just as we use real numbers. So we define addition and multiplication between them to be the same as addition and multiplication for real numbers.

We also have clear common-sense notions about what it means to multiply any physical quantity—for example, mass, relative position, angular momentum, or volume—by a real number. We therefore define the product sA between a scalar s and any multivector A to correspond to these common-sense notions. For example, if A is a vector then sA is defined to be the vector that represents the physical displacement s times as long and in the same direction as the displacement represented by A.

B. Generalization of addition

We all know what addition and multiplication between real numbers means, but we're about to invent generalizations of these notions so it is worthwhile to consider what qualities we want to retain.

First consider addition. If we add beans together, we can imagine tossing them together into a sack. To find their sum, we simply take them all out. The order of this physical addition makes no difference: *addition is commutative.* The same imagery of tossing objects into a bag can be used for adding vectors. The sum is simply the result of pulling all the vectors out and successively applying them, independent of the order. The sum $\mathbf{c}=\mathbf{a}+\mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ may be self-evident to any physicist, but we may explicitly define it as the vector $\mathbf{c}$ that represents the phys-

ical displacement resulting from successive applications of the displacements represented by **a** and by **b**. Of course the result of successive physical displacements is independent of the order. (See Refs. 5, 14, and 15 for developments of the geometric algebra of curved spacetime to describe general relativity, for which this is not true.)

We must be careful in defining the addition of bivectors since we want addition to be commutative while addition of the physical archetype, rotations, depends on the order in which the rotations are performed. But very small rotations are commutative, so we can define the sum $C=A+B$ of bivectors A and B in terms of physical rotations in the limit of very small magnitudes: it is that bivector representing the single rotation that is equivalent to rotation A followed by rotation B. If A and B are *not* small, we still can define their sum by scaling A and B down, making the correspondence with rotations, and then scaling the result back up; that is, we use the distributive law discussed later to write

$$A+B=N(1/N)(A+B)=N(A/N+B/N), \tag{1}$$

where N is some number large enough that A/N and B/N are very small, and then use the physical definition of the sum of small bivectors A/N and B/N. The bivector $A+B$ is then well defined, but not equal to the bivector representing physical rotation A followed by rotation B.

The fundamental definitions above for the addition of vectors and of bivectors are both motivated by corresponding physical actions, the addition of displacements and the addition of small rotations. In both cases it is easier to think about addition by visualizing graphic algorithms that can be shown to be equivalent to the physical algorithms. The graphic algorithm for adding vectors is second nature to any physicist: (1) represent the vectors **a** and **b** by directed line segments (arrows), (2) connect them head-to-tail, and (3) draw a directed line segment from the unconnected tail to head to represent the sum $\mathbf{a}+\mathbf{b}$.

The sum of any two bivectors A and B may be visualized by a similar algorithm: (1) represent the bivectors A and B by oriented parallelograms chosen so that they share a common vector factor; that is, $A=\mathbf{a}\wedge\mathbf{c}$ and $B=\mathbf{b}\wedge\mathbf{c}$ for some vectors **a**, **b**, and **c** where is **c** parallel to the line of intersection of the bivector planes; (2) connect the vectors **a** and **b** head-to-tail; (3) draw a parallelogram using the directed line segments representing the sum $\mathbf{a}+\mathbf{b}$ and the common factor **c**. See Fig. 3 below. This algorithm also, incidentally, graphically represents the distributive law, $\mathbf{a}\wedge\mathbf{c}+\mathbf{b}\wedge\mathbf{c}=(\mathbf{a}+\mathbf{b})\wedge\mathbf{c}$, discussed below.

We will use the imagery of a sum as an unordered collection of physical objects when thinking about sums of multivectors of other grades. We will even take this idea one step further and allow sums of multivectors of different grades, such as a real number plus a vector. This addition doesn't "mix" grades; they're simply collected together under one name. This sort of sum of objects of different type is akin to complex numbers, sums of real and imaginary numbers.

An arbitrary multivector may be a sum of scalars, vectors, bivectors, and trivectors. A multivector of a single grade—an n-vector for $n=0$, 1, 2, or 3—will be called a *simple* multivector.

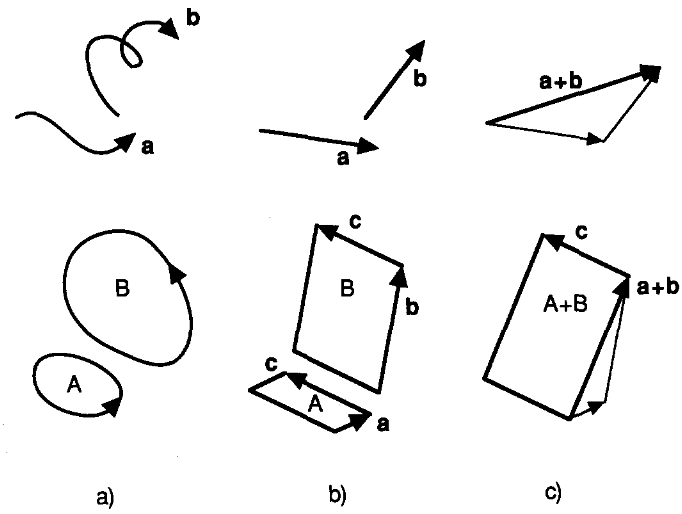

Fig. 3. Addition of vectors and of bivectors. (a) Cartoons of physical displacements **a** and **b** and oriented areas A and B. (b) Standard graphic representations: arrows and oriented parallelograms. (c) Graphic representations of the sum of vectors **a** and **b**, and of bivectors A and B.

C. Features desired in generalizations of multiplication

Now let's look at multiplication. We already assumed (Sec. IV A) the common-sense meaning of multiplying a physical quantity such as a length or an area by a real number. We're now interested in defining more general kinds of products between arbitrary simple multivectors A and B of any grades. For this general discussion we will indicate a product with an asterisk, e.g., $A*B$. There is one very fundamental property that holds for all common uses of the word "product:" a product is *bilinear* in its two arguments, obeying

$$(rA)*(sB)=rs(A*B), \tag{2}$$

for any real numbers r and s. This is obviously true if A and B are also real numbers, and is also true for all other physical products such as the product of two lengths giving an area. This is so fundamental to our notion of what "product" means that we will require it to hold for any more general product between multivectors A and B.

We also want to be able to combine addition and multiplication in familiar ways. The most important property relating addition and multiplication of real numbers is that multiplication is *distributive* with respect to addition:

$$A*(B+C)=A*B+A*C. \tag{3}$$

This is also so fundamental to our notions of addition and multiplication that we will require it of any generalizations.

With these thoughts in mind we now consider possible products between various elements of the algebra.

D. The outer product between any two multivectors

There is no single, obvious definition of a product between two vectors. We want to invent or choose a simple physical or geometrical construction to correspond to a product of any two vectors **a** and **b**. Perhaps the simplest possibility is to attach the vectors **a** and **b** head to tail and use them to define an oriented parallelogram. We will identify this product with the bivector $\mathbf{a}\wedge\mathbf{b}$ in our system and

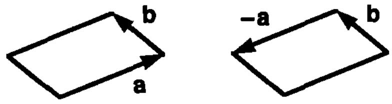

Fig. 4. Graphic representation of $\mathbf{a}\wedge\mathbf{b}=\mathbf{b}\wedge(-\mathbf{a})$.

call it the *outer product* or *wedge product* of **a** and **b**. A bivector may be denoted by a single character, such as "B" or "σ," possibly with a curved line over it as with "$\hat{\sigma}$" to indicate a correspondence with rotation.

In agreement with the physical interpretation of a bivector, the exact pair of vectors **a** and **b** will not matter: if another pair of vectors $\mathbf{a}'$ and $\mathbf{b}'$ describes a different parallelogram but with the same planar orientation and area, we will say that the outer product $\mathbf{a}'\wedge\mathbf{b}'$ equals $\mathbf{a}\wedge\mathbf{b}$. This is analogous to the fact that any scalar can be written in many ways as a product: e.g., $20 = 2*10 = 4*5 = 60*(1/3)$. The graphic representation by a parallelogram is very useful, but one must be careful not to attach significance to the details of any particular parallelogram representing a particular bivector. Recall that the archetypal physical interpretation of a bivector is a discrete rotation, in this case one such that physical lines parallel to **a** rotate toward lines parallel to **b** in the $\mathbf{a}-\mathbf{b}$ plane. Of course there is not a unique pair of vectors that describe this, although there is a unique magnitude and planar orientation.

This geometrical interpretation implies that the outer product between vectors is anticommutative since the parallelogram and orientation of $\mathbf{a}\wedge\mathbf{b}$ equal those of $\mathbf{b}\wedge(-\mathbf{a})$, which equals $-(\mathbf{b}\wedge\mathbf{a})$ by the requirement of bilinearity. See Fig. 4.

Gibbs' vector product $\mathbf{c}=\mathbf{a}\times\mathbf{b}$ is closely related to the outer product: it is a vector perpendicular to $\mathbf{a}\wedge\mathbf{b}$ and with the same magnitude, $|\mathbf{c}|=|\mathbf{a}\wedge\mathbf{b}|$. The precise algebraic relation between them will be given in Sec. V.

This idea of an outer product between vectors has a natural generalization to an outer product between a vector **a** and bivector $\mathbf{b}\wedge\mathbf{c}$: we will identify the wedge or outer product $\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})$ with a volume represented by the parallelpiped described by the three vectors **a**, **b**, and **c**. Again we abstract only the notions of magnitude and direction from this geometrical construct to be represented by this outer product. Only the volume and "direction"—the handedness, right or left, of the vector triplet—matters. The details of the shape are unimportant. We will identify this triple outer product as the trivector $\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})$. Its magnitude is equal to the triple scalar product of conventional vector algebra, $|\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})|=|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$, as can be easily shown algebraically with the results of Sec. V.

The geometrical interpretation also implies that the volume associated with the triple wedge product is independent of how the factors are grouped. This implies that the outer product may be either associative or anti-associative (the same volume but possibly with a sign change). We have no motive to complicate things with anti-associativity, so we will define the sign to be independent of how the factors are grouped and to depend only on the handedness of the vector triplet as they are ordered in the product. Then the outer product between vectors is associative:

$$\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})=(\mathbf{a}\wedge\mathbf{b})\wedge\mathbf{c}. \tag{4}$$

The general idea is that an outer product between a vector **a** and a simple multivector A of any grade corresponds to a new geometrical object with dimension greater than A's by one. Objects with dimension greater than three do not exist in a three-dimensional space, so we define the outer product of a vector with a trivector to be zero. We might consider this outer product to create a 4-D object with a hyper-volume equal to zero since the vector has no component perpendicular to the trivector.[7] We define the outer product $\mathbf{a}\wedge r$ of a vector **a** with a scalar r to be ordinary multiplication of the vector by the scalar, resulting in the vector $r\mathbf{a}$.

The pattern of symmetry under exchange of operands for this outer product between a vector **a** and different multivectors is worth noting: It is symmetric for scalars r,

$$\mathbf{a}\wedge r\equiv\mathbf{a}r=r\mathbf{a}\equiv r\wedge\mathbf{a}, \tag{5}$$

antisymmetric for vectors **b**,

$$\mathbf{a}\wedge\mathbf{b}=-\mathbf{b}\wedge\mathbf{a}, \tag{6}$$

and symmetric for any bivector $A=\mathbf{b}\wedge\mathbf{c}$:

$$\begin{aligned}\mathbf{a}\wedge A=\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})&=(\mathbf{a}\wedge\mathbf{b})\wedge\mathbf{c}=-(\mathbf{b}\wedge\mathbf{a})\wedge\mathbf{c}\\&=-\mathbf{b}\wedge(\mathbf{a}\wedge\mathbf{c})=+\mathbf{b}\wedge(\mathbf{c}\wedge\mathbf{a})\\&=+(\mathbf{b}\wedge\mathbf{c})\wedge\mathbf{a}=A\wedge\mathbf{a}.\end{aligned} \tag{7}$$

That is, the symmetry alternates with grade. These last three equations may be summarized by the single equation,

$$\mathbf{a}\wedge M=(-1)^{g}M\wedge\mathbf{a}, \tag{8}$$

where **a** is any vector and M is any simple multivector with grade g.

E. The inner product between vectors

Now we continue to consider other possible products between two vectors. Another simple quantity we can define given two unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ is the *inner product* $\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}$, equal to the length of the projection of either one onto the other. This can be immediately generalized to nonunit vectors by the fundamental requirement (2) to give a definition of the inner product $\mathbf{a}\cdot\mathbf{b}$ of nonunit vectors $\mathbf{a}=a\hat{\mathbf{a}}$ and $\mathbf{b}=b\hat{\mathbf{b}}$ in terms of the inner product between the unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ and the magnitudes $a=|\mathbf{a}|$ and $b=|\mathbf{b}|$ of the vectors:

$$\mathbf{a}\cdot\mathbf{b}=ab(\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}). \tag{9}$$

This is of course the same as Gibbs' scalar or dot product in conventional vector algebra.

The symmetry of this product is important:

$$\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}. \tag{10}$$

This follows immediately from the symmetry of the inner product between unit vectors according to the geometrical definition, and the commutativity of the scalar magnitudes a and b.

F. Generalization of the inner product to other multivectors via the associative geometric product

This system is now not very different from conventional vector algebra. Two of the mathematical objects—scalars

and vectors—are exactly the same as in vector algebra, while the other two—bivectors and trivectors—correspond to pseudo-vectors and pseudo-scalars. One of the products, the inner product between vectors, is exactly the same as the dot product of conventional vector algebra. The other type of product, the outer or wedge product between two or three vectors, corresponds to the cross product between two vectors and to the triple scalar product between three vectors in conventional vector algebra.

Although the wedge product is defined between any pair of multivectors, the inner product has so far been defined only between vectors. We will not be satisfied with this omission and will continue until all products are defined. By exploiting our freedom to define the remaining inner products we can create a unified system with great power, as follows.

Noting that the two products between vectors have opposite symmetry, we define a general *geometric product* $\mathbf{ab}$ between two vectors $\mathbf{a}$ and $\mathbf{b}$ as the *sum* of the inner and outer product:

$$\mathbf{ab}=\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\wedge\mathbf{b}. \tag{11}$$

This product is a sum of a scalar and a bivector, but recalling that we use similar sums of different objects routinely when using complex numbers, we don't let this bother us.

Yet more complicated, you might say. But this is actually the start of a dramatic unification and simplification. This algebra of multivectors with the operations of addition and the single geometric product is beginning to look like simply the algebra of real numbers. We will see very soon that this unusual geometric product actually simplifies abstract algebraic manipulations, while the inner and outer products from which it is composed continue to hold their separate geometric significances. This is quite analogous to the simplification that results from treating complex numbers as single entities with a single product rather than as ordered pairs of real numbers having a complicated, 2-component product.

We henceforth consider this geometric product to be the algebraically fundamental product rather than the inner and outer products separately. The geometric product is not generally commutative,

$$\mathbf{ab}\neq\mathbf{ba}, \tag{12}$$

unless it happens that $\mathbf{a}\wedge\mathbf{b}=0$ ($\mathbf{a}$ parallel to $\mathbf{b}$) so that $\mathbf{ab}=\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}=\mathbf{ba}$; nor is it anticommutative,

$$\mathbf{ab}\neq-\mathbf{ba}, \tag{13}$$

unless it happens that $\mathbf{a}\cdot\mathbf{b}=0$ ($\mathbf{a}$ perpendicular to $\mathbf{b}$) so that $\mathbf{ab}=\mathbf{a}\wedge\mathbf{b}=-\mathbf{b}\wedge\mathbf{a}=-\mathbf{ba}$. We're stuck with this complication. But from this product the inner and outer products can easily be extracted because of their opposite symmetry:

$$\mathbf{a}\cdot\mathbf{b}=\frac{\mathbf{ab}+\mathbf{ba}}{2} \tag{14}$$

and

$$\mathbf{a}\wedge\mathbf{b}=\frac{\mathbf{ab}-\mathbf{ba}}{2}. \tag{15}$$

We have defined all outer products, but we haven't yet defined the inner product between a vector and a bi- or tri-vector. It would be very convenient if we could extract these inner and outer products from the geometric product in a similar way. We will therefore require that, whatever these inner products are, *the symmetry of the inner product between a vector and any simple multivector must be opposite that of the corresponding outer product.* We already have the symmetry of the outer product between any vector $\mathbf{a}$ and any other simple multivector M, given by Eq. (8); the symmetry of the inner product must then be given by the corresponding equation

$$\mathbf{a}\cdot M=-(-1)^{g}M\cdot\mathbf{a}. \tag{16}$$

By using symmetries (8) and (16), the inner and outer products can be easily extracted from the geometric products $\mathbf{a}M$ and $M\mathbf{a}$ between any vector $\mathbf{a}$ and simple multivector M of grade g:

$$\mathbf{a}\cdot M=\frac{\mathbf{a}M-(-1)^{g}M\mathbf{a}}{2} \tag{17}$$

and

$$\mathbf{a}\wedge M=\frac{\mathbf{a}M+(-1)^{g}M\mathbf{a}}{2}. \tag{18}$$

If M is a vector so that $g=1$ or trivector so that $g=3$, these are the same as (14) and (15); if M is a bivector, the signs differ.

One immediate consequence of symmetry (17) is that the inner product of a vector a with a scalar r ($g=0$) is antisymmetric:

$$\mathbf{a}\cdot r=-r\cdot\mathbf{a}. \tag{19}$$

But we are assuming that our scalars behave just like conventional real numbers in any kind of product; in particular, they commute with everything, including vectors in an inner product:

$$\mathbf{a}\cdot r=r\cdot\mathbf{a}. \tag{20}$$

The only possible product for operands that both anticommute and commute is the number zero, so we must have

$$\mathbf{a}\cdot r=0=r\cdot\mathbf{a}, \tag{21}$$

for all vectors $\mathbf{a}$ and real numbers r. The geometric product $r\mathbf{a}$ therefore equals the outer product $r\wedge\mathbf{a}$ between a scalar and a vector and is the same as the conventional product of a scalar and vector.

The symmetry requirements (17) and (18) constrain but do not define the remaining inner products. A property extremely desirable for ease of use in any algebra is associativity, so we will exploit our present freedom by requiring that for any three multivectors A, B, and C, *the geometric product is associative,*

$$A(BC)=(AB)C. \tag{22}$$

One may worry with some justification that this will not be possible. After all, we have a strange geometric product that results in a disconcerting sum of dissimilar things like real numbers and plane segments, and we are now requiring that in a multiple product the mixed jumble of scalar, vector, bivector, and trivector objects be independent of the order of operation. However, it turns out to be just restrictive enough: *The requirement of associativity uniquely determines all remaining inner products between elements of the algebra.*

Before looking at some details of this, let's look at why associativity is so important. Associativity so permeates typical algebraic manipulations of real numbers that one doesn't normally give it a second thought. The products in Gibbs' vector algebra, on the other hand, are not generally associative. For example,

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\neq(\mathbf{a}\times\mathbf{b})\times\mathbf{c}. \tag{23}$$

But it's worse than this: not only are products in conventional vector algebra nonassociative, but many are not even defined! For example, we can't even ask if the product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ is associative because $(\mathbf{a}\cdot\mathbf{b})\times\mathbf{c}$ is not defined. Similarly we can't ask if $\mathbf{a}\cdot\mathbf{b}\cdot\mathbf{c}$ is associative, because neither $(\mathbf{a}\cdot\mathbf{b})\cdot\mathbf{c}$ nor $\mathbf{a}\cdot(\mathbf{b}\cdot\mathbf{c})$ are defined.

To have all products defined and to have the algebraically fundamental geometric product associative makes many algebraic manipulations easy that are not even possible in Gibbs' vector algebra, as the application to rotations will show. But perhaps the most profound consequence of this is that we can define derivatives and integrals just as they are defined for real functions of real variables, as long as we are careful to maintain the order of factors. The resulting calculus has an unmatched power and unity, as the application to electrodynamics in the following paper will show.

We will now use the associativity of the geometric product to deduce an expression for the inner product $\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})$ between a vector $\mathbf{a}$ and a bivector $\mathbf{b}\wedge\mathbf{c}$ in terms of inner and outer products already defined. The basic idea of the derivation is to write the expression solely in terms of geometric products and then commute factors until terms cancel. This is done by repeatedly using the definition of the inner product between vectors, written as $\mathbf{ab}=2\mathbf{a}\cdot\mathbf{b}-\mathbf{ba}$, and the associativity of the geometric product. The proof is really quite simple, but explicitly writing out the details may be helpful:

$$\begin{aligned}
\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c}) &= \frac{\mathbf{a}(\mathbf{b}\wedge\mathbf{c})-(\mathbf{b}\wedge\mathbf{c})\mathbf{a}}{2} \quad \text{(definition of inner product)}\\
&= \frac{\mathbf{a}(\mathbf{bc}-\mathbf{cb})-(\mathbf{bc}-\mathbf{cb})\mathbf{a}}{4} \quad \text{(definition of outer product)}\\
&= \frac{(\mathbf{ab})\mathbf{c}-(\mathbf{ac})\mathbf{b}-(\mathbf{bc}-\mathbf{cb})\mathbf{a}}{4} \quad \text{(associativity)}\\
&= \frac{(2\mathbf{a}\cdot\mathbf{b}-\mathbf{ba})\mathbf{c}-(2\mathbf{a}\cdot\mathbf{c}-\mathbf{ca})\mathbf{b}-(\mathbf{bc}-\mathbf{cb})\mathbf{a}}{4} \quad \text{(definition of inner product)}\\
&= \frac{(2\mathbf{a}\cdot\mathbf{b})c-\mathbf{b}(\mathbf{ac})-(2\mathbf{a}\cdot\mathbf{c})\mathbf{b}+\mathbf{c}(\mathbf{ab})-(\mathbf{bc}-\mathbf{cb})\mathbf{a}}{4} \quad \text{(associativity)}\\
&= \frac{(2\mathbf{a}\cdot\mathbf{b})c-\mathbf{b}(2\mathbf{a}\cdot c-\mathbf{ca})-(2\mathbf{a}\cdot\mathbf{c})\mathbf{b}+\mathbf{c}(2\mathbf{a}\cdot\mathbf{b}-\mathbf{ba})-(\mathbf{bc}-\mathbf{cb})\mathbf{a}}{4} \quad \text{(definition of inner product)}\\
&= \frac{(2\mathbf{a}\cdot\mathbf{b})\mathbf{c}-(2\mathbf{a}\cdot\mathbf{c})\mathbf{b}+(\mathbf{bc})\mathbf{a}-(2\mathbf{a}\cdot\mathbf{c})\mathbf{b}+(2\mathbf{a}\cdot\mathbf{b})\mathbf{c}-(\mathbf{cb})\mathbf{a}-(\mathbf{bc}-\mathbf{cb})\mathbf{a}}{4} \quad \text{(associativity)}\\
&= (\mathbf{a}\cdot\mathbf{b})\mathbf{c}-(\mathbf{a}\cdot\mathbf{c})\mathbf{b} \quad \text{(additive cancellation of terms and multiplicative cancellation of factors).}
\end{aligned} \tag{24}$$

One can verify that this product $\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})$ is a vector in the direction of $\mathbf{a}$ projected onto the plane defined by $\mathbf{b}\wedge\mathbf{c}$ and then rotated by 1/4 turn in the plane. Although this product may at first look like it would have limited use, it is extremely useful in dealing with rotations and the Lorentz force law, among other things. It is in fact equal to (minus) the triple cross product $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$, as comparison of the final line of (24) with the "bac minus cab" rule for the triple product will show.

The outer product of a vector with a multivector raises the grade of the multivector by one; reviewing the results for the inner product of a vector with a scalar, a vector, and a bivector reveals the corresponding pattern that the inner product of a vector with a multivector lowers the grade by one. Both the outer product of a vector with a trivector and the inner product of a vector with a scalar result in objects outside of our algebra by these rules and are defined to equal zero.

Calculations similar to (24) allow all remaining inner products—vector · trivector, bivector · trivector, trivector · trivector, and bivector · bivector—to be written as sums of multivectors with coefficients involving only inner products between vectors. In this way all products are defined in terms of simpler, previously defined quantities.

V. SUMMARY OF BASIC FEATURES OF THE ALGEBRA AND ITS GEOMETRICAL INTERPRETATION

Geometric algebra has been developed constructively in the past sections, starting with desired details and building generalities. Now that we have this algebra sketched out it is valuable to stand back and look at the general character of the system as a whole.

We have an algebra of multivectors with one kind of addition and one general kind of multiplication, the geometric product. Except for multiplication being noncom-

mutative, multivectors can be manipulated on paper just as if they were real numbers: we can simplify expressions using the distributive and associative laws, divide, factor out common factors in equations, etc.

This simplicity is joined with a complex structure induced by the noncommutivity of the geometric product: symmetric and antisymmetric parts of the product are distinguished as inner and outer products, and four qualitatively distinct classes of elements, simple multivectors of grades 0, 1, 2, and 3, belong to the algebra.

This complexity, including both the multiplicity of classes of elements and the various inner and outer products between them, corresponds in detail to useful geometrical notions that are easily visualized. Rather than a burden, then, this complexity is a richness that allows us to algebraically represent geometric ideas which, although complex, we consider simple and take for granted because of the human mind's remarkable ability to conceive and reason geometrically.

The symmetric and antisymmetric parts of the geometric product are the quantities with prime geometric or physical significance, while the composite geometric product is the fundamental algebraic operation having the extremely useful property of associativity. We therefore pictorially or graphically understand equations in geometric algebra by visualizing the inner and outer products separately, but rely heavily on the combined geometric product for algebraic manipulations.

VI. MISCELLANEOUS RESULTS

A handful of results and definitions that are useful in practical calculations are given below. Many are used in the application to rotations and in the following paper on geometric calculus and electromagnetism.

(a) We often know that two vectors are either parallel or perpendicular. The geometric product between them then equals either the inner or outer product, respectively; the factors commute in the first case and anticommute in the second. In this case a general technique is to write their product as a geometric product if we want to use associativity, or as an inner or outer product if we want to commute them with no more complication than a possible sign change.

(b) Given any vector r and any unit vector $\mathbf{u}$, we may decompose r into vector components parallel and perpendicular to $\mathbf{u}$ by using the fact that the product of any unit vector with itself is unity ($\mathbf{uu}=\mathbf{u}\cdot\mathbf{u}=1$), using the associativity of the geometric product, and expanding the geometric product in terms of an inner and outer product:

$$\begin{aligned}\mathbf{r}=\mathbf{r}(1)=\mathbf{r}(\mathbf{uu})&=(\mathbf{ru})\mathbf{u}\\&=(\mathbf{r}\cdot\mathbf{u}+\mathbf{r}\wedge\mathbf{u})\mathbf{u}\\&=(\mathbf{r}\cdot\mathbf{u})\mathbf{u}+(\mathbf{r}\wedge\mathbf{u})\mathbf{u}.\end{aligned}\tag{25}$$

According to (a), the first term on the right, $(\mathbf{r}\cdot\mathbf{u})\mathbf{u}$, commutes with $\mathbf{u}$ and the second, $(\mathbf{r}\wedge\mathbf{u})\mathbf{u}$, anticommutes with $\mathbf{u}$:

$$\mathbf{ru}=[(\mathbf{r}\cdot\mathbf{u})\mathbf{u}+(\mathbf{r}\wedge\mathbf{u})\mathbf{u}]\mathbf{u}=\mathbf{u}[(\mathbf{r}\cdot\mathbf{u})\mathbf{u}-(\mathbf{r}\wedge\mathbf{u})\mathbf{u}].\tag{26}$$

(c) We give the special name i to the unit trivector: for any right-handed set of orthonormal vectors $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$,

$$i\equiv\hat{\mathbf{a}}\wedge\hat{\mathbf{b}}\wedge\hat{\mathbf{c}}=\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}}.\tag{27}$$

This is independent of the particular choice of the orthonormal vectors. That is, if $(\hat{\mathbf{a}},\hat{\mathbf{b}},\hat{\mathbf{c}})$ and $(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$ are any two right-handed sets of orthonormal vectors, then

$$i=\hat{\mathbf{a}}\wedge\hat{\mathbf{b}}\wedge\hat{\mathbf{c}}=\hat{\mathbf{x}}\wedge\hat{\mathbf{y}}\wedge\hat{\mathbf{z}}.\tag{28}$$

We can therefore always choose to represent i with the most convenient set of orthonormal vectors. This cannot be proved from prior results, but rather can be taken as an algebraic specification of the nature of our 3D space.

(d) It immediately follows that all triple outer products differ only by their magnitude and sign: for any vectors $\mathbf{u},\mathbf{v},\mathbf{w}$,

$$\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w}=ri,\tag{29}$$

for some real number r such that $|r|=|\mathbf{u}\wedge\mathbf{v}\wedge\mathbf{w}|$.

(e) The unit trivector has the important property,

$$i^2=-1.\tag{30}$$

We can show this by writing i as the product of three orthonormal vectors $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$ and then using the associativity of the geometric product and (a) above:

$$\begin{aligned}i^2=(\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}})(\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}})&=(\hat{\mathbf{a}}\hat{\mathbf{b}})(\hat{\mathbf{c}}\hat{\mathbf{a}})(\hat{\mathbf{b}}\hat{\mathbf{c}})\\&=(\hat{\mathbf{a}}\wedge\hat{\mathbf{b}})(\hat{\mathbf{c}}\wedge\hat{\mathbf{a}})(\hat{\mathbf{b}}\wedge\hat{\mathbf{c}})\\&=---(\hat{\mathbf{b}}\wedge\hat{\mathbf{a}})(\hat{\mathbf{a}}\wedge\hat{\mathbf{c}})(\hat{\mathbf{c}}\wedge\hat{\mathbf{b}})\\&=-(\hat{\mathbf{b}}\hat{\mathbf{a}})(\hat{\mathbf{a}}\hat{\mathbf{c}})(\hat{\mathbf{c}}\hat{\mathbf{b}})\\&=-\hat{\mathbf{b}}(\hat{\mathbf{a}}\hat{\mathbf{a}})(\hat{\mathbf{c}}\hat{\mathbf{c}})\hat{\mathbf{b}}=-\hat{\mathbf{b}}\hat{\mathbf{b}}=-1.\end{aligned}\tag{31}$$

One may easily verify that any unit bivector also enjoys this property. Any subalgebra consisting of the scalars and either all trivectors or all bivectors associated with one particular planar orientation is therefore isomorphic to the algebra of complex numbers.

(f) The unit pseudoscalar i times any vector $\mathbf{c}$ equals a bivector $i\mathbf{c}$ perpendicular to $\mathbf{c}$, with magnitude equal to $c=|\mathbf{c}|$. This is easily verified by writing i as the product of three orthonormal vectors $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$ with the last one chosen to be parallel to $\mathbf{c}$ so that $\mathbf{c}=c\hat{\mathbf{c}}$:

$$i\mathbf{c}=(\hat{\mathbf{a}}\hat{\mathbf{b}}\hat{\mathbf{c}})(\hat{\mathbf{c}}c)=(\hat{\mathbf{a}}\hat{\mathbf{b}})(\hat{\mathbf{c}}\hat{\mathbf{c}})c=(\hat{\mathbf{a}}\wedge\hat{\mathbf{b}})c.\tag{32}$$

From the geometric interpretation of $i\mathbf{c}$ it is obvious that $\mathbf{c}$'s projection onto $i\mathbf{c}$ is zero, or

$$\mathbf{c}\cdot(i\mathbf{c})=0,\tag{33}$$

and also that the parallelpiped defined by $\mathbf{c}$ and $i\mathbf{c}$ has volume $|\mathbf{c}|^2$, or

$$\mathbf{c}\wedge(i\mathbf{c})=i|\mathbf{c}|^2.\tag{34}$$

These last two equations can alternatively be easily proved algebraically using (39) below with $\mathbf{a}=B=\mathbf{c}$.

(g) The cross product $\mathbf{c}=\mathbf{a}\times\mathbf{b}$ of Gibbs' vector algebra is a vector perpendicular to the outer product $\mathbf{a}\wedge\mathbf{b}$ for any vectors $\mathbf{a}$ and $\mathbf{b}$. Using the last result we can show that these products are related by a simple factor of i:

$$i(\mathbf{a}\times\mathbf{b})=\mathbf{a}\wedge\mathbf{b}.\tag{35}$$

We may take this as a *definition* of the cross product; multiplying (35) by $-i$, we equivalently have

$$\mathbf{a}\times\mathbf{b}=-i(\mathbf{a}\wedge\mathbf{b}). \tag{36}$$

(h) The pseudoscalar i commutes with all vectors and therefore all multivectors. This may be proved using a) and c) above.

(i) We can always separately write the scalar, vector, bivector, or trivector parts of any equation in geometric algebra, just as we can always separately write the real and imaginary parts of any equation in the algebra of complex numbers. As an example we will derive two useful identities. Consider the equation

$$i(\mathbf{a}B)=(i\mathbf{a})B, \tag{37}$$

for any vector $\mathbf{a}$ and any simple multivector B of any grade. This is simply a statement of the associativity of the geometric product in the particular case of one factor being the unit trivector i. Explicitly writing the geometric products involving B as inner and outer products and then using the distributive law on the left, we have

$$i(\mathbf{a}\cdot B)+i(\mathbf{a}\wedge B)=(i\mathbf{a})\wedge B+(i\mathbf{a})\cdot B. \tag{38}$$

If B is a vector, this equation has trivector and vector parts; if B is a bivector, it has bivector and scalar parts. In any case, the two parts of this equation are

$$i(\mathbf{a}\wedge B)=(i\mathbf{a})\cdot B \quad \text{and} \quad i(\mathbf{a}\cdot B)=(i\mathbf{a})\wedge B. \tag{39}$$

These identities can be very useful for simplifying algebraic expressions.

(j) Conventional vector algebra is a subsystem of geometric algebra and all of its results apply. For example, any vector $\mathbf{v}$ may be expanded in any set of orthonormal vectors $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, and $\hat{\mathbf{c}}$ as

$$\mathbf{v}=(\mathbf{v}\cdot\hat{\mathbf{a}})\hat{\mathbf{a}}+(\mathbf{v}\cdot\hat{\mathbf{b}})\hat{\mathbf{b}}+(\mathbf{v}\cdot\hat{\mathbf{c}})\hat{\mathbf{c}}. \tag{40}$$

(k) We define the *reverse* $A^\dagger$ of any multivector A as that expression obtained from A by reversing the order of all vector factors in all simple multivectors making up A. Since the outer product of vectors is anticommutative, all bivectors and trivectors change sign upon reversion,

$$(\mathbf{a}\wedge\mathbf{b})^\dagger\equiv\mathbf{b}\wedge\mathbf{a}=-\mathbf{a}\wedge\mathbf{b} \tag{41}$$

and

$$(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})^\dagger\equiv\mathbf{c}\wedge\mathbf{b}\wedge\mathbf{a}=-\mathbf{c}\wedge(\mathbf{a}\wedge\mathbf{b})$$
$$=+(\mathbf{a}\wedge\mathbf{c})\wedge\mathbf{b}=-\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}), \tag{42}$$

including of course the unit trivector,

$$i^\dagger=-i. \tag{43}$$

Scalars and vectors, on the other hand, are unchanged. It immediately follows from the definition that the reverse of a geometric product of multivectors is the reverse of the reversed factors:

$$(AB...CD)^\dagger=D^\dagger C^\dagger...B^\dagger A^\dagger. \tag{44}$$

This definition is especially useful in the description of rotations where pairs of factors, one the reverse of the other, frequently occur. Reversion corresponds to Hermitian conjugation in matrix algebra.

VII. APPLICATION TO ROTATIONS

In this section a description of discrete rotations is developed. As an example of its use, the inner product between an arbitrarily rotated arbitrary vector and a second arbitrary vector is calculated. The use of geometric algebra in numerical computations of rotations is discussed and compared with the conventional matrix approach.

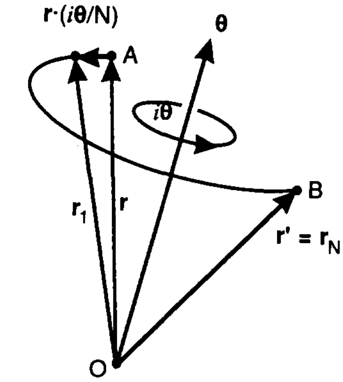

Fig. 5. The rotation of a particle from point A, represented relative to point O by the vector $\mathbf{r}$, to point B, represented by the vector $\mathbf{r}'$. This can be considered as a sum of N small rotations and corresponding displacements. The first displacement is (approximately) the vector $\mathbf{r}\cdot(i\boldsymbol{\theta}/N)$.

A. The representation of a rotation in geometric algebra

Consider a point on some physical body. Let the position of the point relative to some specified reference point be represented by the vector r. Suppose the body is rotated about the reference point with the paths of all points parallel to some plane and with some magnitude represented by the orientation and magnitude of the bivector $i\boldsymbol{\theta}$, where $\boldsymbol{\theta}$ is some vector parallel to the axis of rotation. After the rotation the point is at some new position represented by the vector $\mathbf{r}'$. See Fig. 5. We will show below that $\mathbf{r}'$ is related to $\mathbf{r}$ and $\boldsymbol{\theta}$ by

$$\mathbf{r}'=e^{-i\theta/2}\mathbf{r}e^{+i\theta/2}. \tag{45}$$

No corresponding simple expression exists in vector algebra. Rather, one typically introduces a set of basis vectors and components in that basis and develops matrix algebra.

Expression (45) can be derived by considering a series of small rotations, each one in the plane $i\boldsymbol{\theta}$ (perpendicular to the axis $\boldsymbol{\theta}$) and of magnitude θ/N where N is some large number. We first recall the result (24) for the inner product $\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})$ of a vector $\mathbf{a}$ with a bivector $\mathbf{b}\wedge\mathbf{c}$, and its geometric interpretation as the projection of $\mathbf{a}$ onto the plane of $\mathbf{b}\wedge\mathbf{c}$ followed by a rotation of 1/4 turn in that plane and multiplication by the magnitude $|\mathbf{b}\wedge\mathbf{c}|$ of the bivector. Applying this to the vector $\mathbf{r}$ and the bivector $i\boldsymbol{\theta}/N$, we can see that the result of one small rotation is approximately

$$\mathbf{r}_1\approx\mathbf{r}+\mathbf{r}\cdot(i\boldsymbol{\theta}/N), \tag{46}$$

with an error of order $(r\theta/N)^2$. We can rewrite the inner product in terms of the geometric product and continue to neglect quadratic terms to write

$$\mathbf{r}_1\approx\mathbf{r}+[\mathbf{r}(i\boldsymbol{\theta}/N)-(i\boldsymbol{\theta}/N)\mathbf{r}]/2$$
$$\approx\left(1-\frac{i\boldsymbol{\theta}/2}{N}\right)\mathbf{r}\left(1+\frac{i\boldsymbol{\theta}/2}{N}\right). \tag{47}$$

Applying N small rotations each of size θ/N gives the vector

$$\mathbf{r}_N \approx \left(1-\frac{i\theta/2}{N}\right)^N \mathbf{r}\left(1+\frac{i\theta/2}{N}\right)^N. \tag{48}$$

With the usual definition of the exponential function,

$$-e^A \equiv \lim_{N\to\infty}\left(1+\frac{A}{N}\right)^N, \tag{49}$$

extended to any multivector A, we see that the Eq. (45) to be proved is just shorthand for the last result, (48), in the limit of very large N and with $\mathbf{r}_N=\mathbf{r}'$.

Let us look at the exponential factor more closely. For multivectors as for real numbers, the definition (49) is equivalent to the usual series definition:

$$e^{+i\theta/2}=1+\frac{i\theta}{2}+\frac{1}{2!}\left(\frac{i\theta}{2}\right)^2+\frac{1}{3!}\left(\frac{i\theta}{2}\right)^3+\cdots. \tag{50}$$

Now recall that any unit bivector, such as $i\hat{\boldsymbol{\theta}}$, behaves like the unit imaginary number in that its square equals -1. We may therefore rearrange terms just as we may in complex analysis to write

$$e^{+i\theta/2}=\left[1-\frac{1}{2!}\left(\frac{\theta}{2}\right)^2+\cdots\right]+(i\hat{\boldsymbol{\theta}})\left[\frac{\theta}{2}-\frac{1}{3!}\left(\frac{\theta}{2}\right)^3+\cdots\right]$$

$$=\cos\left(\frac{\theta}{2}\right)+(i\hat{\boldsymbol{\theta}})\sin\left(\frac{\theta}{2}\right). \tag{51}$$

The exponential function of a bivector is therefore equal to the sum of a scalar and a bivector.

It is convenient to define the Euler vector,

$$\boldsymbol{\beta}\equiv\hat{\boldsymbol{\theta}}\sin(\theta/2) \tag{52}$$

and the Euler scalar

$$\alpha\equiv\cos(\theta/2), \tag{53}$$

allowing us to write

$$e^{i\theta/2}=\alpha+i\boldsymbol{\beta}. \tag{54}$$

It is also convenient to define the *spinor* R,

$$R\equiv e^{+i\theta/2} \tag{55}$$

In general, a "spinor" is defined as any element of the even subalgebra; that is, any sum of a scalar and bivector. These correspond to the spinors used in quantum mechanics.[5,6] Note that the spinor R is *unimodular*:

$$R^\dagger R=e^{-i\theta/2}e^{+i\theta/2}=(\alpha-i\boldsymbol{\beta})(\alpha+i\boldsymbol{\beta})=\alpha^2+\boldsymbol{\beta}^2=1. \tag{56}$$

The last equality in (56) also makes it explicit that although the sign of α is significant, its magnitude is redundant if we have $\boldsymbol{\beta}$.

With these definitions and the definition of reversion in Sec. V, we may write the rotated vector $\mathbf{r}'$ of (45) in any of the equivalent forms,

$$\mathbf{r}'=R^\dagger\mathbf{r}R=e^{-i\theta/2}\mathbf{r}e^{+i\theta/2}=(\alpha-i\boldsymbol{\beta})\mathbf{r}(\alpha+i\boldsymbol{\beta}). \tag{57}$$

B. The calculation of inner products

Expressions such as (45) may express a geometrical meaning, but all physical measurements yield magnitudes of scalars. These are often inner products between vectors. For example, a Cartesian component of a vector is the inner product of the vector with a basis vector, and the matrix elements of a rotation are the inner products of basis vectors with rotated basis vectors. To calculate such inner products one typically would either draw a picture and graphically figure out an expression involving sines and cosines, or use matrices and Cartesian components. With geometric algebra we can alternatively calculate such quantities entirely algebraically and often with significantly less effort than required by matrix algebra.

As an example of algebraic evaluation of an inner product in terms of other measurable quantities (inner products) we will find the inner product $\mathbf{s}\cdot\mathbf{r}'$ of the rotated vector $\mathbf{r}'$ with an arbitrary second vector $\mathbf{s}$, in terms of inner products between $\mathbf{s}$, the unrotated vector $\mathbf{r}$, and the Euler scalar α and vector $\boldsymbol{\beta}$ specifying the rotation. Using (57) this is

$$\mathbf{s}\cdot\mathbf{r}'=\mathbf{s}\cdot[(\alpha-i\boldsymbol{\beta})\mathbf{r}(\alpha+i\boldsymbol{\beta})]. \tag{58}$$

The general plan will be to move the vectors $\mathbf{r}$ and $\mathbf{s}$ together. We first use (25) to write $\mathbf{r}$ as a sum of vectors $\mathbf{r}_\parallel$ and $\mathbf{r}_\perp$ that are parallel and perpendicular to the rotation axis $\hat{\boldsymbol{\theta}}$ or equivalently to $\hat{\boldsymbol{\beta}}=\hat{\boldsymbol{\theta}}$:

$$\mathbf{r}=\mathbf{r}_\parallel+\mathbf{r}_\perp=\hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\beta}}\cdot\mathbf{r})+\hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\beta}}\wedge\mathbf{r}). \tag{59}$$

Vectors parallel to $\boldsymbol{\beta}$, such as $\mathbf{r}_\parallel$, commute with $\boldsymbol{\beta}$, while vectors perpendicular to $\boldsymbol{\beta}$, such as $\mathbf{r}_\perp$, anticommute. Substituting (59) into (58) therefore yields

$$\mathbf{s}\cdot\mathbf{r}'=\mathbf{s}\cdot[(\alpha-i\boldsymbol{\beta})(\alpha+i\boldsymbol{\beta})\mathbf{r}_\parallel+(\alpha-i\boldsymbol{\beta})^2\mathbf{r}_\perp]$$

$$=\mathbf{s}\cdot[\mathbf{r}_\parallel+(\alpha-i\boldsymbol{\beta})^2\mathbf{r}_\perp]. \tag{60}$$

Writing out the squared factor and distributing the inner product gives

$$\mathbf{s}\cdot\mathbf{r}'=\mathbf{s}\cdot[\mathbf{r}_\parallel+(\alpha^2-\beta^2-2i\alpha\boldsymbol{\beta})\mathbf{r}_\perp]$$

$$=\mathbf{s}\cdot\mathbf{r}_\parallel+(\alpha^2-\beta^2)\mathbf{s}\cdot\mathbf{r}_\perp-\mathbf{s}-(2i\alpha\boldsymbol{\beta}\mathbf{r}_\perp). \tag{61}$$

The first term is easily evaluated using the expression implied by (59) for $\mathbf{r}_\parallel$ to get

$$\mathbf{s}\cdot\mathbf{r}_\parallel=\mathbf{s}\cdot\hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\beta}}\cdot\mathbf{r}). \tag{62}$$

The second term may be evaluated most easily by first evaluating the expression for $\mathbf{r}_\perp$ implied by (59) in terms of inner products, using expression (24) for the triple product:

$$\mathbf{r}_\perp=\hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\beta}}\wedge\mathbf{r})$$

$$=\hat{\boldsymbol{\beta}}\cdot(\hat{\boldsymbol{\beta}}\wedge\mathbf{r})=(\hat{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{\beta}})\mathbf{r}-(\hat{\boldsymbol{\beta}}\cdot\mathbf{r})\hat{\boldsymbol{\beta}}=\mathbf{r}-(\hat{\boldsymbol{\beta}}\cdot\mathbf{r})\hat{\boldsymbol{\beta}}. \tag{63}$$

The second term is then

$$\mathbf{s}\cdot(\alpha^2-\beta^2)\mathbf{r}_\perp=(\alpha^2-\beta^2)\mathbf{s}\cdot[\mathbf{r}-(\hat{\boldsymbol{\beta}}\cdot\mathbf{r})\hat{\boldsymbol{\beta}}]$$

$$=(\alpha^2-\beta^2)[(\mathbf{s}\cdot\mathbf{r})-(\mathbf{s}\cdot\hat{\boldsymbol{\beta}})(\hat{\boldsymbol{\beta}}\cdot\mathbf{r})]. \tag{64}$$

The third term of (61) may be quickly evaluated by substituting the expression for $\mathbf{r}_\perp$ from the first equality of (63) and using the associativity of the geometric product and identity (39); we get

$$-\mathbf{s}\cdot[2i\alpha\beta\mathbf{r}_\perp] = \mathbf{s}\cdot[2i\alpha\beta\hat{\beta}(\mathbf{r}\wedge\hat{\beta})]$$
$$= \mathbf{s}\cdot[2i\alpha\hat{\beta}\hat{\beta}(\mathbf{r}\wedge\boldsymbol{\beta})]$$
$$= 2\alpha|\hat{\beta}|^2\mathbf{s}\cdot[i(\mathbf{r}\wedge\boldsymbol{\beta})] = 2\alpha i(\mathbf{r}\wedge\boldsymbol{\beta}\wedge\mathbf{s}). \tag{65}$$

Combining (61), (62), (64), and (65) we can write the desired inner product as

$$\mathbf{s}\cdot\mathbf{r}' = (\mathbf{s}\cdot\hat{\beta})(\mathbf{r}\cdot\hat{\beta}) + (\alpha^2-\beta^2)[(\mathbf{s}\cdot\mathbf{r}) - (\mathbf{s}\cdot\hat{\beta})(\mathbf{r}\cdot\hat{\beta})]$$
$$+2\alpha i(\mathbf{r}\wedge\boldsymbol{\beta}\wedge\mathbf{s})$$
$$= (\alpha^2-\beta^2)(\mathbf{s}\cdot\mathbf{r}) + (1+\beta^2-\alpha^2)(\mathbf{s}\cdot\hat{\beta})(\mathbf{r}\cdot\hat{\beta})$$
$$+2\alpha i(\mathbf{r}\wedge\boldsymbol{\beta}\wedge\mathbf{s}). \tag{66}$$

This may be simplified using the relation $\alpha^2+\beta^2=1$ to give the final result,

$$\mathbf{s}\cdot\mathbf{r}' = (2\alpha^2-1)(\mathbf{s}\cdot\mathbf{r}) + 2(\mathbf{s}\cdot\boldsymbol{\beta})(\mathbf{r}\cdot\boldsymbol{\beta})$$
$$+2\alpha i(\mathbf{r}\wedge\boldsymbol{\beta}\wedge\mathbf{s}). \tag{67}$$

This is probably the most convenient simple expression for this inner product. The last term, representing volume of a parallelpiped, may alternatively be written in terms of the more familiar cross product:

$$i(\mathbf{r}\wedge\boldsymbol{\beta}\wedge\mathbf{s}) = i\{[i(\mathbf{r}\times\boldsymbol{\beta})]\wedge\mathbf{s}\}$$
$$= i\{i[(\mathbf{r}\times\boldsymbol{\beta})\cdot\mathbf{s}]\} = -(\mathbf{r}\times\boldsymbol{\beta})\cdot\mathbf{s}. \tag{68}$$

C. Numerical computations and rotation matrices

Sometimes it is convenient or necessary to use Cartesian components, such as for numerical computations. For example, suppose the vector $\mathbf{r}$ has components r_j with respect to the orthonormal basis vectors $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$, and $\boldsymbol{\sigma}_3$ so that $\mathbf{r}$ may be written

$$\mathbf{r} = \sum_j \boldsymbol{\sigma}_j r_j, \quad r_j \equiv \boldsymbol{\sigma}_j\cdot\mathbf{r}. \tag{69}$$

Using the same basis vectors the Euler vector may be written

$$\boldsymbol{\beta} = \sum_j \boldsymbol{\sigma}_j \beta_j, \quad \beta_j \equiv \boldsymbol{\sigma}_j\cdot\boldsymbol{\beta}. \tag{70}$$

We may naturally ask, what are the components r_i' of the vector

$$\mathbf{r}' = \sum_i \boldsymbol{\sigma}_i r_i', \quad r_i' = \boldsymbol{\sigma}_i\cdot\mathbf{r}', \tag{71}$$

resulting from rotating $\mathbf{r}$ according to $\boldsymbol{\beta}$?

An expression for these components is

$$r_i' = \boldsymbol{\sigma}_i\cdot\mathbf{r}' = \boldsymbol{\sigma}_i\cdot(R^\dagger\mathbf{r}R)$$
$$= \boldsymbol{\sigma}_i\cdot\left[R^\dagger\left(\sum_j \boldsymbol{\sigma}_j r_j\right)R\right] = \sum_j [\boldsymbol{\sigma}_i\cdot(R^\dagger\boldsymbol{\sigma}_j R)]r_j, \tag{72}$$

or

$$r_i' = \sum_j M_{ij} r_j, \quad \text{where } M_{ij} \equiv \boldsymbol{\sigma}_i\cdot(R^\dagger\boldsymbol{\sigma}_j R). \tag{73}$$

The matrix element M_{ij} is identical to the inner product evaluated in the last section with $\mathbf{s}=\boldsymbol{\sigma}_i$ and $\mathbf{r}=\boldsymbol{\sigma}_j$. Then using expression (67) we have

$$M_{ij} = (2\alpha^2-1)\delta_{ij} + 2\beta_i\beta_j + 2\alpha i\left[\sum_k (\boldsymbol{\sigma}_i\wedge\boldsymbol{\sigma}_j\wedge\boldsymbol{\sigma}_k)\beta_k\right], \tag{74}$$

where $\beta_i \equiv \boldsymbol{\beta}\cdot\boldsymbol{\sigma}_i$ are the Cartesian components of $\boldsymbol{\beta}$ relative to the chosen basis. The usual totally antisymmetric tensor (or Levi-Civita symbol) may be defined by

$$\epsilon_{ijk} \equiv -i(\boldsymbol{\sigma}_i\wedge\boldsymbol{\sigma}_j\wedge\boldsymbol{\sigma}_k). \tag{75}$$

For example, if $(i,j,k)=(1,2,3)$ this is $\epsilon_{123} = -ii = +1$. With this definition, we may rewrite (74) as

$$M_{ij} = (2\alpha^2-1)\delta_{ij} + 2\beta_i\beta_j - 2\alpha\left(\sum_k \epsilon_{ijk}\beta_k\right). \tag{76}$$

This expresses a rotation matrix in terms of the vector $\boldsymbol{\theta}$ that specifies the axis and magnitude of the rotation, with α and $\boldsymbol{\beta}$ given by the simple definitions (53) and (52). $\boldsymbol{\theta}$ generally has direct physical significance and so is easy to identify in a real problem. Euler angles, in comparison, are often difficult to find in real problems.

Describing rotations with the Euler vector and scalar rather than with matrices or Euler angles also leads to a simple expression for the composition of rotations as follows. Let $R=\alpha+i\boldsymbol{\beta}$ be the composite R_aR_b of two rotations, $R_a=\alpha_a+i\boldsymbol{\beta}_a$ followed by $R_b=\alpha_b+i\boldsymbol{\beta}_b$:

$$\alpha+i\boldsymbol{\beta} = (\alpha_a+i\boldsymbol{\beta}_a)(\alpha_b+i\boldsymbol{\beta}_b)$$
$$= (\alpha_a\alpha_b - \boldsymbol{\beta}_a\cdot\boldsymbol{\beta}_b) + i[\alpha_b\boldsymbol{\beta}_a + \alpha_a\boldsymbol{\beta}_b + i(\boldsymbol{\beta}_a\wedge\boldsymbol{\beta}_b)]. \tag{77}$$

Explicitly writing the scalar and bivector parts of (77), we have

$$\alpha = \alpha_a\alpha_b - \boldsymbol{\beta}_a\cdot\boldsymbol{\beta}_b \tag{78}$$

and

$$\boldsymbol{\beta} = \alpha_b\boldsymbol{\beta}_a + \alpha_a\boldsymbol{\beta}_b + i(\boldsymbol{\beta}_a\wedge\boldsymbol{\beta}_b) = \alpha_b\boldsymbol{\beta}_a + \alpha_a\boldsymbol{\beta}_b - \boldsymbol{\beta}_a\times\boldsymbol{\beta}_b. \tag{79}$$

The corresponding component equations are straightforward.

With these results we may numerically calculate, for example, the matrix corresponding to two or more successive rotations by first finding the resulting Euler scalar α and vector $\boldsymbol{\beta}$ using (78) and (79), and then using (76) to calculate the corresponding matrix.

Numerically evaluating the expression (76) for the matrix in terms of the Euler vector and scalar requires 13 low-precision multiplications, and numerical evaluation of the product of two rotations, Eqs. (78) and (79), requires 15 low-precision multiplications. In contrast, numerically evaluating a rotation matrix in terms of Euler angles requires many high-precision multiplications and evaluations of trigonometric functions, and numerical evaluation of a product of two rotation matrices requires 27 high-precision multiplications. High precision is needed to maintain orthogonality of the matrix; this is automatically guaranteed by the Euler vector and scalar as long as the scalar α is periodically corrected to ensure that $\alpha^2 = 1-|\boldsymbol{\beta}|^2$.

Suppose, for example, we are numerically integrating a rotation (perhaps specified in real time by input from a

mouse or joystick) and displaying a rotated figure on a computer screen. Using geometric algebra, we would constantly update the Euler scalar and vector using (78) and (79) with R_a equal to the most recent total integrated rotation and R_b equal to the most recent new incremental rotation. Whenever necessary, perhaps with every update, we would adjust α to ensure that $\alpha^2=1-|\beta|^2$ and then calculate the rotation matrix using (76). This requires 29 low-precision products.

In contrast, a conventional algorithm based on Euler angles and matrices requires approximately twice as many multiplications plus several trigonometric evaluations, all high precision. The required CPU time can be many times greater than that required using algorithms based on geometric algebra.

VIII. AN APPLICATION: A FREELY ROTATING CYLINDRICALLY SYMMETRIC BODY

Perhaps the greatest strength of geometric algebra is in its extension to geometric calculus. All results of the calculus of real numbers apply to the calculus of multivector functions of a single scalar variable such as time, with the proviso that the order of factors be maintained unless they are known to commute. This calculus will be used in this section without any further development.

The time derivative of expressions such as (45) when either or both $\mathbf{s}$ and $\boldsymbol{\theta}$ depend on time can be easily calculated. As already pointed out, an expression like (45) does not even exist in conventional vector algebra. This cripples the analysis of rotating bodies and rotating reference frames. Largely because of this deficiency, treatments of these topics using vector algebra are generally more difficult and confusing than necessary.

In this section we illustrate the use of geometric calculus in physical problems involving rotations by solving for the motion of a freely rotating cylindrically symmetric rigid body. We closely follow the treatment by Hestenes.[6]

Consider a freely spinning, axially symmetric body. By "solving for the motion" we mean finding the spinor R as a function of time that gives the position $\mathbf{r}(t)$ according to (45) of any point on the body, given its initial position $\mathbf{s}$ (all positions are relative to the center of mass). We will do this by finding one equation relating R to the angular velocity, writing the angular velocity in terms of the angular momentum and the body's symmetry axis, and then solving this equation for $R=R(t)$.

First we must ask, what do we *mean* by "angular velocity"? One way to formulate a definition is to consider the change in orientation of the spinning object from time t to a short time later, $t+dt$. When we say that $\boldsymbol{\omega}$ is the angular velocity at time t, we mean that the plane of rotation at that moment is defined by the bivector $i\boldsymbol{\omega}$ (or equivalently, for our three-dimensional space, that the axis of rotation is defined by the vector $\boldsymbol{\omega}$) and that the magnitude of the rotation during a short time interval dt is ωdt. Then the rotation of the object from time t to time $t+dt$ is described by the spinor

$$S\equiv e^{i\boldsymbol{\omega}dt/2}\approx 1+i\boldsymbol{\omega}dt/2. \tag{80}$$

Both $\boldsymbol{\omega}$ and S generally depend on time, but since we need S only at some particular time t we have not explicitly indicated this dependence.

We would like to relate this angular velocity to the time-dependent rotation operator $R(t)$ that characterizes the object's orientation for all t. The spinor-valued function $R(t)$ may be expanded in a Taylor series about any particular time t just as any real-valued function may be; i.e.,

$$R(t+dt)\approx R(t)+\dot{R}(t)dt, \tag{81}$$

where $\dot{R}(t)$ is the time derivative of $R(t)$. Let $\mathbf{s}$ be the position of an arbitrary point on the object at time $t=0$. The position at time $t+dt$ may be written either as $R^\dagger(t+dt)\mathbf{s}R(t+dt)$ or equivalently as $S^\dagger R^\dagger(t)\mathbf{s}R(t)S$. Equating these and using approximations (80) and (81), we have

$$(R^\dagger+\dot{R}^\dagger dt)\mathbf{s}(R+\dot{R}dt)$$
$$\approx(1-i\boldsymbol{\omega}dt/2)R^\dagger\mathbf{s}R(1+i\boldsymbol{\omega}dt/2). \tag{82}$$

Taking the limit of a very small time interval dt, this implies

$$R^\dagger\mathbf{s}\dot{R}+\dot{R}^\dagger\mathbf{s}R=R^\dagger\mathbf{s}Ri\boldsymbol{\omega}/2-i\boldsymbol{\omega}R^\dagger\mathbf{s}R/2. \tag{83}$$

Comparing terms we see that this is true if

$$\dot{R}=Ri\boldsymbol{\omega}/2 \tag{84}$$

and

$$\dot{R}^\dagger=-i\boldsymbol{\omega}R^\dagger/2. \tag{85}$$

Equation (85) is just the reversion of (84) and so is redundant. We therefore take (84) as our primary relation between $\dot{R}$ and $\boldsymbol{\omega}$.

It is worth noting that we *cannot* simply take the derivative of $R(t)=\exp[i\boldsymbol{\theta}(t)]$ to write it as $\exp(i\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$, since $\boldsymbol{\theta}$ and $\dot{\boldsymbol{\theta}}$ may not be in the same direction and therefore may not commute. This may be easiest to understand by recalling the expression for the exponential function as the Taylor series given in (50); then the derivative of the $\boldsymbol{\theta}(t)\boldsymbol{\theta}(t)$ term already gives us complications since it equals $\dot{\boldsymbol{\theta}}\boldsymbol{\theta}+\boldsymbol{\theta}\dot{\boldsymbol{\theta}}$. The $\dot{\boldsymbol{\theta}}$ factor can be moved to the right, giving $2\boldsymbol{\theta}\dot{\boldsymbol{\theta}}$, only if $\boldsymbol{\theta}(t)$ equals a constant vector times some function of t; i.e., only if the axis of rotation is fixed. We can, though, complete a similar calculation after expressing R in terms of Euler parameters. We will not need this and so will not pursue this here.

We now rewrite $\boldsymbol{\omega}$ in terms of the angular momentum $\mathbf{L}$ and will then solve (84) for $R(t)$.

Let the body's symmetry axis be parallel to the unit vector $\hat{\mathbf{a}}$. $\hat{\mathbf{a}}$ is fixed in the body, so it changes with time as the body undergoes free motion. Let the moment of inertia about $\hat{\mathbf{a}}$ be called I_a and the moment of inertia about any axis perpendicular to $\hat{\mathbf{a}}$ be called I. The angular momentum $\mathbf{L}$ about the center of mass may be projected into components parallel and perpendicular to $\hat{\mathbf{a}}$:

$$\mathbf{L}=I_a(\boldsymbol{\omega}\cdot\hat{\mathbf{a}})\hat{\mathbf{a}}+I(\boldsymbol{\omega}\wedge\hat{\mathbf{a}})\hat{\mathbf{a}}. \tag{86}$$

Substituting $\boldsymbol{\omega}\wedge\hat{\mathbf{a}}=\boldsymbol{\omega}\hat{\mathbf{a}}-\boldsymbol{\omega}\cdot\hat{\mathbf{a}}$ (definition of the geometric product) into (84) and using $(\boldsymbol{\omega}\hat{\mathbf{a}})\hat{\mathbf{a}}=\boldsymbol{\omega}(\hat{\mathbf{a}}\hat{\mathbf{a}})=\boldsymbol{\omega}$ (associativity of the geometric product), we have

$$\mathbf{L}=I\boldsymbol{\omega}+(I_a-I)(\boldsymbol{\omega}\cdot\hat{\mathbf{a}})\hat{\mathbf{a}}. \tag{87}$$

We can invert this equation, finding an expression for $\boldsymbol{\omega}$ in terms of $\mathbf{L}$ and $\hat{\mathbf{a}}$, by dotting this equation with $\hat{\mathbf{a}}$ and substituting the resulting expression for $\boldsymbol{\omega}\cdot\hat{\mathbf{a}}$ back into the equation we started with. This gives

$$\boldsymbol{\omega}=\frac{1}{I}\left(\mathbf{L}+\frac{I-I_a}{I}(\mathbf{L}\cdot\hat{\mathbf{a}})\hat{\mathbf{a}}\right). \tag{88}$$

A free body has no torque on it so $\mathbf{L}$ is constant. All time dependence in this expression is therefore in the direction $\hat{\mathbf{a}}$ of the symmetry axis. We don't know yet what this time dependence is—that's what we're trying to find—but it is given by (45) just like the position of any particle on the body. If we let $\hat{\boldsymbol{\sigma}}$ be a fixed unit vector in the direction of the symmetry axis before any motion begins, $\hat{\mathbf{a}}$ is given in terms of $\hat{\boldsymbol{\sigma}}$ by

$$\hat{\mathbf{a}}=R^\dagger\hat{\boldsymbol{\sigma}}R. \tag{89}$$

Substituting (89) into the last factor $\hat{\mathbf{a}}$ in (88) and then the result into (84) gives

$$\dot{R}=R\frac{i\mathbf{L}}{2I}+\frac{I-I_a}{I}\frac{i(\mathbf{L}\cdot\hat{\mathbf{a}})}{I_a}\hat{\boldsymbol{\sigma}}R. \tag{90}$$

With the definitions

$$\boldsymbol{\omega}_1=\frac{I-I_a}{I_a}\frac{\mathbf{L}\cdot\hat{\mathbf{a}}}{I}\hat{\boldsymbol{\sigma}} \tag{91}$$

and

$$\boldsymbol{\omega}_2\equiv\frac{\mathbf{L}}{I}, \tag{92}$$

(90) assumes the simpler appearance

$$\dot{R}=R\left(\frac{i}{2}\boldsymbol{\omega}_2\right)+\left(\frac{i}{2}\boldsymbol{\omega}_1\right)R. \tag{93}$$

At this point we know that $\hat{\mathbf{a}}$ depends on time, and so we might expect $\boldsymbol{\omega}_1$ to depend on time. But we will seek a trial solution with the tentative assumption that the projection $\mathbf{L}\cdot\hat{\mathbf{a}}$ of $\hat{\mathbf{a}}$ onto $\mathbf{L}$ is time independent, so that $\boldsymbol{\omega}_1$ is a constant. We will find a solution and it will turn out that $\mathbf{L}\cdot\hat{\mathbf{a}}$ is in fact time independent as provisionally assumed.

With this assumption, (93) is a simple first order differential equation with constant coefficients. The solution $R(t)$ has the usual exponential form, augmented appropriately due to the noncommutivity of the geometric product:

$$R=e^{i/2\boldsymbol{\omega}_1 t}e^{i/2\boldsymbol{\gamma}}e^{i/2\boldsymbol{\omega}_2 t}, \tag{94}$$

where $\exp(i\boldsymbol{\gamma}/2)$ is any unimodular spinor parametrized by the constant vector $\boldsymbol{\gamma}$. We can verify that (94) is a solution to (93) by taking the time derivative of (94) with the help of the usual product rule for derivatives, being careful not to change the order of factors unless we're sure they commute:

$$\begin{aligned}\dot{R}&=\frac{d}{dt}(e^{i/2\boldsymbol{\omega}_1 t}e^{i/2\boldsymbol{\gamma}}e^{i/2\boldsymbol{\omega}_2 t})\\
&=\left(\frac{d}{dt}e^{i/2\boldsymbol{\omega}_1 t}\right)(e^{i/2\boldsymbol{\gamma}})(e^{i/2\boldsymbol{\omega}_2 t})+(e^{i/2\boldsymbol{\omega}_1 t})\left(\frac{d}{dt}e^{i/2\boldsymbol{\gamma}}\right)(e^{i/2\boldsymbol{\omega}_2 t})+(e^{i/2\boldsymbol{\omega}_1 t})(e^{i/2\boldsymbol{\gamma}})\left(\frac{d}{dt}e^{i/2\boldsymbol{\omega}_2 t}\right)\\
&=\frac{i}{2}\boldsymbol{\omega}_1(e^{i/2\boldsymbol{\omega}_1 t})(e^{i/2\boldsymbol{\gamma}})(e^{i/2\boldsymbol{\omega}_2 t})+0+(e^{i/2\boldsymbol{\omega}_1 t})(e^{i/2\boldsymbol{\gamma}})(e^{i/2\boldsymbol{\omega}_2 t})\frac{i}{2}\boldsymbol{\omega}_2\\
&=\frac{i}{2}\boldsymbol{\omega}_1 R+R\frac{i}{2}\boldsymbol{\omega}_2.\end{aligned} \tag{95}$$

We define the factors

$$R_1\equiv e^{+i\boldsymbol{\omega}_1 t/2},\quad R_\gamma\equiv e^{+i\boldsymbol{\gamma}/2},\quad\text{and}\quad R_2\equiv e^{+i\boldsymbol{\omega}_2 t/2}, \tag{96}$$

so that we may write the spinor R as

$$R=R_1R_\gamma R_2. \tag{97}$$

The path $\mathbf{r}=\mathbf{r}(t)$ of any arbitrary point on the body initially at $\mathbf{s}$ may now be written

$$\begin{aligned}\mathbf{r}=R^\dagger\mathbf{s}R&=(R_2^\dagger R_\gamma^\dagger R_1^\dagger)\mathbf{s}(R_1R_\gamma R_2)\\
&=R_2^\dagger[R_\gamma^\dagger(R_1^\dagger\mathbf{s}R_1)R_\gamma]R_2,\end{aligned} \tag{98}$$

where we have again used the associativity of the geometric product.

Equation (98) has a simple physical interpretation: To find the orientation of the object at any particular time t,

(1) start with the object in its orientation prior to any motion (represented by an arbitrary point $\mathbf{s}$ on the body),

(2) rotate the object about its symmetry axis $\hat{\boldsymbol{\sigma}}$ by the angle $\omega_1 t$ as described by R_1,

(3) tilt the object about the fixed axis $\boldsymbol{\gamma}$ by the angle γ as described by R_γ (this represents an arbitrary initial condition), and lastly

(4) precess the rotated, tilted object about the fixed axis of angular momentum $\mathbf{L}$ by the angle $\omega_2 t$ as described by R_2.

Applying this solution to the initial symmetry axis $\hat{\boldsymbol{\sigma}}$ (a fixed vector) to find the symmetry axis $\hat{\mathbf{a}}\equiv R^\dagger\hat{\boldsymbol{\sigma}}R$ during the motion, we see that $\mathbf{L}\cdot\hat{\mathbf{a}}$ is indeed constant as assumed.

The reader is encouraged to compare this simple algebraic result and its straightforward physical interpretation with a conventional description using vector analysis and Cartesian components.

IX. CONCLUDING REMARKS

In this short tutorial we have only been able to sketch some basic ideas of geometric algebra and apply them to an elementary problem from classical mechanics. Many other topics in mechanics are analyzed more easily, concisely, or thoroughly with geometric algebra. And no analysis ever

need be more difficult or complicated, since conventional vector algebra is a subsystem of geometric algebra.

Geometric algebra may be naturally extended in several useful ways that have not been considered here: (a) A calculus may be developed that contains conventional vector calculus as a subsystem but has additional expressive power and unity. Electromagnetism may be described extremely concisely with this calculus. These topics are outlined in a following paper.[10] (See Ref. 16 for another treatment with a different approach.) (b) We may begin with four-dimensional spacetime rather than three-dimensional space and develop the geometric algebra and calculus of spacetime.[5] Electromagnetism is described with great clarity and simplicity using this spacetime calculus, and the Dirac equation is seen in a new light.[5,17,18] Both classical electromagnetic fields and Dirac spinor fields in spacetime may be naturally projected onto a spacelike 3D hyperplane to return to descriptions in terms of the geometric algebra of three-dimensional space. (c) Geometric algebra may be naturally extended to describe curved spacetime and general relativity.[5,14,15]

A hierarchy of geometric algebras exists that unifies normally disparate topics in physics, as follows. (1) The geometric algebra of spacetime is isomorphic to the algebra of Dirac matrices. (2) The even subalgebra of this (defined as the algebra of the 0-, 2-, and 4-vectors) is isomorphic to the geometric algebra of three-dimensional space (2-vectors in spacetime containing a time-like vector factor are identified with vectors in space, while 2-vectors in spacetime made of two spacelike vector factors are identified with bivectors in space), and to the algebra of the Pauli matrices. (3) The subalgebra of this consisting of the scalars and all bivectors of space is isomorphic to Hamilton's quaternions. (4) Finally, any subalgebra of scalars and multiples of any one particular bivector is isomorphic to complex numbers, as is the subalgebra of all scalars and pseudoscalars (see Ref. 19 for a recent discussion of this). Nature has chosen to allow many things to be described by a unified system of geometric algebra and calculus that we physicists normally describe with a potpourri of disjoint mathematical systems.

After reading the many definitions and unconventional developments described in this paper one may reasonably question whether the advantages of geometric algebra are really worth the trouble. In response I urge caution in comparing the relative merits of this unfamiliar system with familiar and widely used systems since familiar systems are easily perceived to be simpler and easier to use than an unfamiliar one, regardless of merits. I hope this tutorial has persuaded the reader that geometric algebra does, indeed, warrant further study.

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Salam is one of those bold thinkers, uninclined to let little formal difficulties get in the way of what might prove to be a good idea. He once offered me the cheerful, if somewhat dubious advice, 'Publish all your ideas—people will only remember those that turn out well.'

John Polkinghorne, *Rochester Roundabout—The Story of High Energy Physics* (W. H. Freeman, New York, 1989), p. 54.