Games of imperfect information and and modal logic

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1 Basic Modal Logic

We start with a class of propositional atoms, $p, q, ...$ In addition, for each positive integer $i < k$ we have available two modal operators

$$\Diamond_i \text{ and } \Box_i$$

which are of the same modality type i. The operators $\Diamond_i$ and $\Box_i$, are said to be duals. Different occurrences of each such operator will be marked by natural numbers as in the following example

$$\Diamond_{1,1}\Diamond_{1,2}\Box_{2,3}\Box_{5,4}p$$

We denote the set of formulas of this modal language by $ML(k)$.

For each modality type i, there will be an accessibility relation R_i. That is, an k-ary modal structure for the modal propositional language L will have the form

$$M = (D, R_0, ..., R_{k-1}, h)$$

where D is a nonempty set of possible worlds, each R_i is a binary relation on D, and h is a function which assigns a subset of D to each propositional atom.

Truth of the sentence φ in M at the point $w \in D$, $M \vDash_w \varphi$, is defined in a straigthforward way (each sentence is in negation normal form):

(a) $M \vDash_w p$ iff $w \in h(p)$
(b) $M \vDash_w \neg p$ iff $w \notin h(p)$
(c) $M \vDash_w \varphi \vee \psi$ iff $M \vDash_w \varphi$ or $M \vDash_w \psi$
(d) $M \vDash_w \Box_i\varphi$ iff for all w' such that $R_i ww' : M \vDash_{w'} \varphi$
(e) $M \vDash_w \Diamond_i\varphi$ iff there is w' such that $R_i ww'$ and $M \vDash_{w'} \varphi$

The interpretation can be given alternatively in terms of a semantic game $G(M, \varphi, w)$ whose moves are straightforward:

(i) If φ is atom p or its negation $\neg p$, then there is no move. If $w \in h(p)$ ($w \in h(\neg p)$), then *Eloise* wins and *Abelard* looses. Otherwise *Abelard* wins and *Eloise* looses.

(ii) If φ is $\psi \vee \theta$ ($\psi \wedge \theta$), then *Eloise* (*Abelard*) picks up $\chi \in \{\varphi, \theta)$ and the game goes on as $G(M, \chi, w)$.

(iii) If φ is $\Box_p$ ($\Diamond_p$), then *Abelard* (*Eloise*) picks up w' such that $R_p ww'$. If there is no such w', *Eloise* (*Abelard*) wins and *Abelard* (*Eloise*) looses. Otherwise the game continues with the remaining formula and the world w'.

The winning strategies of the players are defined in the same way as before. Then we have game-theoretical truth and falsity:

$$M \models^+_{GTS,w} \varphi \text{ iff there is a winning strategy for } Eloise \text{ in } G(M, \varphi, w)$$

$$M \models^-_{GTS,w} \varphi \text{ iff there is a winning strategy for } Abelard \text{ in } G(M, \varphi, w)$$

It is straightforward to check, by a double induction, that the two accounts are equivalent (assuming the axiom of choice), i.e.

$$M \vDash_w \varphi \Leftrightarrow M \models^+_{GTS,w} \varphi$$

and

$$M \vDash_w \neg\varphi \Leftrightarrow M \models^-_{GTS,w} \varphi$$

We shall now extend the present setting in order to allow for more dependencies and independencies between different modality operators. In the new logic, called $IFML(k)$, the set of well formed formulas will have the form:

$$IFML = \{\varphi : \varphi \in ML(k)\} \cup \{O_1...O_{n-1}(O_n/W)\varphi : \varphi \in ML(k), n \geq 1\}$$

where:

- Each O_p is one of the modal operators $\Diamond_i, \Box_i$.
- W is a (possibly empty) set of natural numbers, $W \subset \{1, ..., n-1\}$.

1.1 Uniformity interpretation

We fix an k-ary modal structure $M = (D, R_0, ..., R_{k-1}, h)$ and define truth and falsity (at a point w) of a sentence φ in this structure via an evaluation game $G(M, \varphi, w)$ which extends the rules given above by the corresponding rule for the formulas of the form $O_1...O_{n-1}(O_n/W)\varphi$. Each operator $O_i, 1 \leq i \leq n$ will prompt a move by *Eloise* or *Abelard* as shown above.

It will help to present the game in extensive normal form, as we did in earlier sections. The game $G_A(M, \varphi, w) = (N, H, Z, P, (u_i)_{i \in N})$ is based on the set of actions

$$A = Dom(M) \cup Sub(\varphi) \cup \{*\}$$

and consists of the following elements:

- $N = \{\exists, \forall\}$

- The set of histories H is defined inductively:

  (i) $(\varphi, w) \in H$

  (ii) If $h \frown (\psi \vee \theta, w') \in H$ $(h \frown (\psi \wedge \theta, w') \in H)$,

  then $h \frown (\psi \vee \theta, w') \frown (\psi, w') \in H$ and $h \frown (\psi \vee \theta, w') \frown (\theta, w') \in H$ $(h \frown (\psi \wedge \theta, w') \frown (\psi, w') \in H$ and $h \frown (\psi \wedge \theta, w') \frown (\theta, w') \in H)$.

- If $h \frown (O_i\psi, w') \in H$, then if O_i is either $\Box_p$ or $\Diamond_p$, $((\Box_p/W)$ or $(\Diamond_p/W))$, then $h \frown (O_i\psi, w') \frown (\psi, w'') \in H$, for every w'' such that $R_p(w', w'')$. If there is no such w'', then $h \frown (O_i\psi, w') \frown (Fail, *) \in H$.

- The definition of the player function $P : H \rightarrow \{\exists, \forall\}$ should be also obvious.

- $Z \subseteq H$ is the set of terminal histories of the game: A history $h = (h_0, \ldots, h_{n-1}) \in H$ is terminal iff there is no h_n of the form (χ, w') such that $h = (h_0, \ldots, h_n) \in H$, or if $h_{n-1} = (Fail, *)$. Thus we notice that every terminal history is either (p, w'), for p an atom or the negation of an atom, or $(Fail, *)$.

- The payoff functions $(u_i)_{i\in\{\exists,\forall\}}$ are defined in the usual way: For any $h = (h_0, \ldots, h_{n-1}) \in Z$,

  $$u_\exists(h) = \begin{cases} 1, \text{ if } h_{n-1} = (p, w') \text{ and } w' \in h(p), \text{ or } h_{n-1} = (Fail, *) \text{ and } P(h_0, \ldots, h_{n-2}) = \forall \\ -1, \text{ otherwise.} \end{cases}$$

  and

  $$u_\forall(h) = \begin{cases} 1, \text{ if } h_{n-1} = (p, w') \text{ and } w' \notin h(p), \text{ or } h_{n-1} = (Fail, *) \text{ and } P(h_0, \ldots, h_{n-2}) = \exists \\ -1, \text{ otherwise.} \end{cases}$$

Thus all the present games are zero-sum games.

For any arbitrary nonterminal history $h = ((\varphi_0, w_0), ..., (\varphi_{n-1}, w_{n-1}))$, we call $(\varphi_0, ...\varphi_{n-1})$ the left projection of h $(pr_l(h))$ and $(w_0, ...w_{n-1})$ its right projection $pr_r(h))$.

We partition the sets $P^{-1}(\exists)$ and $P^{-1}(\forall)$ into sets of equivalence classes determined by the following equivalence relations.

Let h, h' be arbitrary nonterminal histories in the set $P^{-1}(\exists)$.

$h \sim_\exists h' \Leftrightarrow$ $h = h'$ and their last labelling formula is not of the form$(O_n/W)\psi$, or their last labelling formula is of the form $(O_n/W)\psi$ and $pr_l(h) = pr_l(h')$, and for all $i \notin W : pr_r(h)$ and $pr_r(h')$ coincide on their i^{th}elements.

The definition of $h \sim_\forall h'$ is completely similar.

Now we obtain the partitions:

$$I_\exists = \{[h]_{\sim_\exists} : h \in P^{-1}(\exists)\}$$

and

$$I_\forall = \{[h]_{\sim_\forall} : h \in P^{-1}(\forall)\}.$$

A strategy for a player $s \in \{\exists, \forall\}$ is any function $f_s \colon P^{-1}(s) \to A$. That is, f_s is defined on all the possible histories $h \in H$ where the player s is supposed to move.

For any possible choices $\bar{c}$ made by *Abelard*, if *Eloise* is making her choices according to a strategy $f_\exists$, then a finite sequence $h_{\bar{c}}$ is formed. The strategy $f_\exists$ is a *winning* one if it satisfies the following conditions:

- Each sequence $h_{\bar{c}}$ is such that $h_{\bar{c}} \in H$ and $u_\exists(h_{\bar{c}}) = 1$
- $f_\exists(h) = f_\exists(h')$ whenever $h \sim_\exists h'$.

The definition of a winning strategy for *Abelard* is similar. In the sequel we shall often leave out mention of the formula components in the arguments of the strategy functions.

Example. We consider the sentence $\Box_{0,1}\Diamond_{1,2}/\{1\}q$ and the model $M = (Q, <, >, h)$ where $(Q, <)$ is the set of rationals with their ordering $<$ and $>$ is the inverse of $<$. We will show that, no matter how h is defined, we have:

$$M \models_0^+ \Box_{0,1}\Diamond_{1,2}/\{1\}q \Leftrightarrow M \models_0^+ q \vee \Diamond_{1,1}q$$

Assume $M \models_0^+ \Box_{0,1}\Diamond_{1,2}/\{1\}q$. This means there is a function f such that for all rational numbers $r < 0$, $f(0,r)$ is constant, $f(0,r) > r$, and $f(0,r) \in h(q)$. Thus $f(0,r)$ can be either 0 or a rational number greater than 0. But then we also have: $M \models_0^+ q \vee \Diamond_{1,1}q$. The other direction is obvious.

The next step is to show that, when h satisfies certain conditions, the game $G(\Box_{0,1}\Diamond_{1,2}/\{1\}q, M, 0)$ is non-determinate, that is, we have both

$$M \not\models_0^+ \Box_{0,1}\Diamond_{1,2}/\{1\}q$$

and

$$M \not\models_0^- \Box_{0,1}\Diamond_{1,2}/\{1\}q$$

Suppose that

(a) there are no non-negative rationals in $h(q)$, and

(b) there are negative rationals arbitrarily near 0 which belong to $h(q)$, but not 0 itself.

Obviously, from the above equivalence it follows that *Eloise* cannot have a winning strategy in the game $G(\Box_{0,1}\Diamond_{1,2}/\{1\}q, M, 0)$. And *Abelard* cannot have a winning strategy either, because there is no rational $r < 0$ such that for all rationals r' greater than r: $r' \in h(q)$.

Example. We fix the sentence φ: $\Box_{0,1}\Box_{1,2}\Diamond_{1,3}/\{1,2\}(q \vee \neg q)$, and the model $M = (Q, >, <, h)$ where $(Q, <)$ is the set of rational with their ordering $<$ and $>$ is the inverse of $<$.

Let us first show that *Eloise* cannot have a winning strategy at 0. We notice that all the sequences of the form

$$\{((\varphi, 0), (\Box_{1,2}\Diamond_{1,3}/\{1,2\}(q \vee \neg q), r), (\Diamond_{1,3}/\{1,2\}(q \vee \neg q), r')) : \ r > 0, r' < r\}$$

have the same left projection, and on the other side, their right projections

$$\{(0, r, r') : r > 0, r' < r\}$$

coincide for all $i \notin \{1,2\}$ (i.e. $i = 0$). Hence they belong to the same information set of *Eloise*. Because r and r' cover the whole domain Q, there cannot be an unique rational number which is smaller than all rationals. Thus there cannot be a uniform legal strategy for *Eloise*. On the other side, *Abelard* cannot have a winning strategy either, because there is no possible world at which the negation of the sentence $(q \vee \neg q)$ would hold. Thus the game is nondetermined.

Example. Let φ be $\Box_{0,1}\Diamond_{1,2}/\{1\}q$, and M the model $M = (Q, <, \prec, h)$, where

- Q is the set of rationals,
- $<$ is the usual ordering of rationals by their magnitudes,
- $\prec$ is the ordering of nonnegative rationals followed by the ordering of the negative rationals
- $h(q) = \{r : r \geq 0\}$.

The first observation is that *Eloise* cannot have a winning strategy in $G(\Box_{0,1}\Diamond_{1,2}/\{1\}q, M, 0)$ given the fact that there cannot be a rational number r which is greater (in the ordering $\prec$) than all rationals and which is such that q holds at r: there is no positive rational which is greater than all the rationals. On the other side, r cannot be a negative rational either, because, even if r is greater (in the ordering $\prec$) than all positive rationals, we still have $r \notin h(q)$.

Abelard cannot have a winning strategy either, because for whichever rational $r > 0$ *Abelard* chooses, *Eloise* can reply with $r' > r$. So we got another example here of the non-determiness of a game.■

It is straightforward to see that the determined fragment of $IFML$ is equivalent with ML. More exactly, let C be a class of modal structures and L_{det} the class of $IFML$-formulas φ such that for every $M \in C$ and every w in the domain of M we have: either *Eloise* or *Abelard* has a winning strategy in $G(\varphi, M, w)$. Then it can be shown that each such φ is equivalent on C with an ML-formula ψ_φ:

Eloise has a winning strategy in $G(\varphi, M, w)$ iff *Eloise* has a winning strategy in $G(\psi_\varphi, M, w)$.

The formula ψ_φ, may be found in a straightforward way: for φ an ML-formula, ψ_φ is φ itself; and for φ of the form $O_1...O_{n-1}(O_n/W)\chi$ with $\chi \in ML$, ψ_φ is $O_1...O_{n-1}O_n\chi$.

Obviously, if *Eloise* has a winning strategy in $G(\varphi, M, w)$ then she can use the same strategy in *Eloise* to win $G(\psi_\varphi, M, w)$. And if *Eloise* does not have a winning strategy in $G(\varphi, M, w)$ then, by the assumption of determinacy, *Abelard* has one, which he can use also to win $G(\psi_\varphi, M, w)$. Hence *Eloise* cannot not have a winning strategy in $G(\psi_\varphi, M, w)$.

1.2 Bisimulations

Let $M = (D, R_0, ..., R_{k-1}, h)$ and $M' = (D', R_0^{'}, ..., R_{k-1}^{'}, h)$ be two k-ary modal structures. A bisimulation between M and $M'$is a binary relation $R_{D,D'} \subseteq D \times D'$ which satisfies the following clauses:

(1) Atomic harmony:

$$B_{D,D'}(d, d') \Rightarrow \text{ For all atomic } p : d \in h(p) \Leftrightarrow d' \in h'(p)$$

(2) Zigzag forwards:

$$B_{D,D'}(d, d') \text{ and } R_i(d, c) \Rightarrow \exists c'(R_i^{'}(d', c') \text{ and } B_{D,D'}(c, c'))$$

(3) Zigzag Backwards:

$$B_{D,D'}(d, d') \text{ and } R_i^{'}(d', c') \Rightarrow \exists c(R_i(d, c) \text{ and } B_{D,D'}(c, c'))$$

Lemma 1 *Bisimulation lemma. Let $B_{D,D'}$ be a bisimulation between two k-ary modal structures $M = (D, R_0, ..., R_{k-1}, h)$ and $M' = (D', R_0^{'}, ..., R_{k-1}^{'}, h)$.. Then for any formula $\varphi \in ML(k)$, and all $d \in D, d' \in D'$:*

$$B_{D,D'}(d, d') \Rightarrow M \models_d^+ \varphi \Leftrightarrow M' \models_{d'}^+ \varphi$$

Proof: By induction on the complexity of φ.

The case for φ an atomic formula ot its negation follows directly from the above clause (1).

Suppose φ has the form $\Box_i\psi$ and that $M \models_d^+ \Box_i\psi$. Hence for any c such that $R_i(d, c)$, *Eloise* wins $G(\psi, M, c)$. Let c' be such that $R_i^{'}(d', c')$. Then by the Backward zigzag, there is c such that $R_i(d, c)$ and $B_{D,D'}(c, c')$. From the assumption $M \models_d^+ \Box_i\psi$, we get $M \models_c^+ \psi$ which together with $B_{D,D'}(c, c')$ and the inductive hypothesis yield $M \models_{c'}^+ \psi$, whence $M \models_{d'}^+ \Box_i\psi$. The converse is similar and so are all the other cases.

We shall now use the above Bissimulation lemma to prove a simple fact about the relation between the logics $ML(1)$ and $IFML(1)$. In particular we will show that the latter is strictly stronger than the former. This will follow from the following claims:

1 There is an $IFML(1)$ formula ψ such that for every model $M = (D, R, h)$ and possible world $w \in D$

$$(D, R, h) \models_w^+ \psi \ \Leftrightarrow R \text{ and } w \text{ stand in a certain relation } P$$

2 There are models $M = (D, R, h)$ and $M' = (D', R', h')$ and possible worlds $w \in D, w' \in D'$ such that

R and w do not stand in the relation P but R' and w' stand in the relation P

3 For every formula $\varphi \in ML(1)$

$$M \models_w^+ \varphi \Leftrightarrow M' \models_{w'}^+ \varphi$$

(1)-(3) jointly entail that there is no $ML(1)$-sentence φ such that for all models $M = (D, R, h)$ and every $w \in D$:

$$M \models_w^+ \varphi \Leftrightarrow R \text{ and } w \text{ stand in the relation } P. \qquad (+)$$

For suppose, for a contradiction, that there is an $ML(1)$-sentence φ such that (+) holds for every model $M = (D, R, h)$ and every world $w \in D$. Fix the models M and M' and the possible worlds w and w' as in (2). From (2) and our assumption it follows that $M' \models_{w'}^+ \varphi$. Hence from (3) we get $M \models_w^+ \varphi$, which combined with our assumption yields that R and w stand in the relation P. But this contradicts (2).

Example 2 *We say that in the model $M = (D, R, h)$ the relation R is ortomodular in $w \in D$, if there is $v \in D$ such that for all $w' \in D : Rww' \Rightarrow Rw'v$. We shall take ortomodularity to be the relation P above. We notice that for every model $M = (D, R, h)$ and $w \in D$ we have*

$$M \models_w^+ \Box_1 \Diamond_2/\{1\}(q \vee \neg q) \Leftrightarrow R \textit{ is ortomodular in } w \qquad (+)$$

(For the left-to-right direction, notice that $f(w, w') = v$, for all w' such that Rww' is the winning strategy for Eloise.)

Now we shall construct two models $M = (D, R, h)$ and $M' = (D', R', h')$ with $w_1 \in D$ and $v_1 \in D'$ such that R' is ortomodular in v_1 but R is not ortomodular in w_1.

Let $M = (D, R, h)$ and $M' = (D', R', h')$ where

$D = \{w_1, ..., w_5\}$, $D' = \{v_1, ..., v_4\}$

$R = \{(w_1, w_2), (w_1, w_3), (w_2, w_4), (w_3, w_5), (w_1, w_4), (w_1, w_5)\} \cup \{(w, w) : w \in D\}$

$R' = \{(v_1, v_2), (v_1, v_3), (v_2, v_4), (v_3, v_4), (v_1, v_4)\} \cup \{(v, v) : v \in D'\}$

$h(q) = \{w_2, ..., w_5\}; h'(q) = \{v_2, v_3, v_4\}$.

It is straightforward to check that the relation

$$B = \{(w_1, v_1), (w_2, v_2), (w_3, v_3), (w_4, v_4), (w_5, v_4)\}$$

is a bisimulation between M and M' such that $(w_1, v_1) \in B$. By the Bisimulation lemma, the possible worlds of these structures which are in the bisimulation relation B cannot be distinguished by any $ML(1)-$formulas.■

Example 3 *We get a nice variant of the earlier example by showing that $IFML(1)$ can distinguish between two isomorphical frames (while $ML(1)$ cannot). Two frames $M = (D, R)$ and $M' = (D', R')$ are isomorphical if there is a bijection $f : D \to D'$ such that for all $w, w' \in D$*

$$R(w, w') \Leftrightarrow R'(f(w), f(w')).$$

We shall now construct two models $M = (D, R, h)$ and $M' = (D', R', h')$ such that

- *The frames (D, R) and (D', R') are isomorphical*
- *There is an $IFML(1)$-sentence χ (obviously different from $\Box_1\Diamond_2/\{1\}(q \vee \neg q)$) and $w_1 \in D$, $v_1 \in D'$ such that*

  $M \models^+_{w_1} \Box_1\Diamond_2/\{1\}q$ *but neither* $M' \models^+_{v_1} \Box_1\Diamond_2/\{1\}q$ *nor* $M' \models^-_{v_1} \Box_1\Diamond_2/\{1\}q$
- *R is ortomodular in w_1 and R' is ortomodular in v_1.*
- *There is a bisimulation relation $B_{D,D'}$ such that $(w_1, v_1) \in B_{D,D'}$.*

  We let $M = (D, R, h)$ and $M' = (D', R', h')$ where

  $D = \{w_1, ..., w_6\}$

  $D' = \{v_1, ..., v_6\}$

  $R = \{(w_1, w_2), (w_1, w_3), (w_2, w_4), (w_2, w_5), (w_3, w_5), (w_3, w_6)\}$

  $R' = \{(v_1, v_2), (v_1, v_3), (v_2, v_4), (v_2, v_5), (v_3, v_5), (v_3, v_6)\}$

  $h(q) = \{w_5\}, h'(q) = \{w_4, w_6\}$. *(Cf. the picture below).*

  Clearly the function $f(w_i)v_i$ is an isomorphism between M and M'. Now we notice that for $\chi = \Box_1\Diamond_2/\{1\}q$ we have

  $M \models^+_{w_1} \Box_1\Diamond_2/\{1\}q$ *but neither* $M' \models^+_{v_1} \Box_1\Diamond_2/\{1\}q$ *nor* $M' \models^-_{v_1} \Box_1\Diamond_2/\{1\}q$.

  That is, the strategy f defined by:

  $$f(w_1, w_2) = f(w_1, w_3) = w_5$$

  is winning for Eloise in $G(\chi, M, w_1)$, but on the other side, there is no uniform function which will give a win for Eloise, neither one for Abelard in the game $G(\chi, M, v_1)$. On the other side, there cannot be a winning strategy for Eloise in $G(\chi, M', v_1)$, since there cannot be a function g such that $g(v_1, v_2) = g(v_1, v_3)$ and in addition $g(v_1, v_2) \in h'(q)$. Neither can Abelard choose a possible world $v \in \{v_2, v_3\}$ such that for any possible choise v' of Eloise we have $v' \in h'(q)$.

  Now we shall build up a bisimulation $B_{D,D'}$ such that $(w_1, v_1) \in B_{D,D'}$. This shows, by the bisimulation lemma, that the world w_1 satisfies the same $ML(1)$ formulas in M as the world v_1 in M'.

  Let $B_{D,D'}$ be $B_{D,D'} = \{(w_1, v_1), (w_2, v_2), (w_3, v_3), (w_2, v_3), (w_3, v_2), (w_4, v_5), (w_6, v_5), (w_5, v_4), (w_5, v_6)\}$

  Checking that $B_{D,D'}$ is a bisimulation is a tedious mechanical task that shall not be run here.■

2 Modal structures linear in two dimensions

In this section we will show that $IFML(2)$ is strictly stronger than $ML(2)$. We will use the same technique as in the preceding section.

We say that in the frame $M = (D, R_1, R_2)$ the relations R_1 and R_2 are ortomodular in $w \in D$ if there is one possible world $v \in D$ such that for all w' with R_1ww': $R_2w'v$.

In the same way as earlier, it is obvious that the sentence $\chi = \Box_{0,1}\Diamond_{1,2}/\{1\}(q \vee \neg q)$ defines ortomodularity in a frame, i.e., for every frame $M = (D, R_1, R_2)$ and every $w \in D$:

$$M \models^+_w \Box_{0,1}\Diamond_{1,2}/\{1\}(q \vee \neg q) \Leftrightarrow R_1 \text{ and } R_2 \text{ are ortomodular in } w.$$

We shall build up two models $M = (D, R_1, R_2, h)$ and $N = (D', R_1^{'}, R_2^{'})$ with two possible worlds $w_1 \in D$ and $v_1 \in D'$ such that

- $R_1^{'}$ and $R_2^{'}$ are ortomodular in v_1 but R_1 and R_2 are not ortomodular in w_1.
- There is a bisimulation relation $E_{D,D'}$ such that $(w_1, v_1) \in E_{D,D'}$.

As before this will show that there is no $ML(2)$-sentence φ such that for all modal frames $M = (D, R_1, R_2)$ and every $w \in D$:

$$M \models^+_w \varphi \Leftrightarrow R_1 \text{ and } R_2 \text{ are ortomodular in } w.$$

Let $M = (Q, <, <, h)$, where

Q is the set of rational numbers, $<$ is the natural ordering of the rationals, and $h(q) = Q$.

Let $B = Q \times \{1\}$, and $(B, <^+, <^+)$ be an isomorphic copy of $(Q, <, <)$. Put

$$N = (A', <_1', <_2', h')$$

where

- $A' = Q \cup B$
- $<_1' = < \cup <^+ \cup A \times B$
- $<_2' = < \cup <^+ \cup B \times A$
- $h'(q) = A'$.

Define the binary relation

$$E_{Q,A'} = Q \times A'$$

We claim that E is a bisimulation between the modal structures M and N.

Atomic harmony follows directly from the definition of E and the definition of h and h'.

The proof of the claims

Zigzag forwards in modality type 1

$$E_{Q,A'}(d, d') \text{ and } (d < c) \Rightarrow \exists c'((d' <_1' c') \text{ and } E_{Q,A'}(c, c'))$$

Zigzag forwards in modality type 2

$$E_{Q,A'}(d,d') \text{ and } (d < c) \Rightarrow \exists c'((c' <_2' d') \text{ and } E_{Q,A'}(c,c'))$$

Zigzag Backwards in modality type 1:

$$E_{Q,A'}(d,d') \text{ and } (d' <_1' c') \Rightarrow \exists c(d < c) \text{ and } E_{Q,A'}(c,c'))$$

Zigzag Backwards in modality type 2:

$$E_{Q,A'}(d,d') \text{ and } d' <_2' c') \Rightarrow \exists c(d < c) \text{ and } E_{Q,A'}(c,c'))$$

is straightforward. As an example, consider Zigzag forwards in modality type 1. Assume $E_{Q,A'}(d,d')$ and $(d < c)$. If $d' \in Q$, then there are infinitely many $c' \in Q$ such that $c' <_2' d'$ and $E_{Q,A'}(c,c')$, so pick up one of them. All the other cases are similar.

Now let fix a point $b \in dom(N) \cap B$ and a point $a \in dom(M)$. We notice that $(a,b) \in E$, and also that the relations $<$ and $<$ cannot be ortomodular in a: otherwise, there must be a constant c such that for all x, $a < x$, we must have $x < c$, which is impossible.

On the other side, $<_1'$and $<_2'$are seen to be ortomodular in b: Fix any element c from the set A, and assume that $b <_1' x$, for arbitrary x. Since $b \in B$, we must also have $x \in B$. But $B \times A \subset <_2'$, hence $x <_2' c$.■

It is easy to see that this result can be generalized to all $k \geq 2$.