Gabriel Sandu

Informational Independent Connectives and Epistemic Logic

1 Extensive games of perfect information: general case

We fix a family of actions A which represents the set of possible choices of the players in a game. A sequence $(a_1, ..., a_n)$ of actions represents the consecutive choices of the players, $a_i \in A$.

An *extensive game* $\mathcal{G}_A$ of perfect information is a tuple $\mathcal{G}_A = (N, H, Z, P, (u_i)_{i \in N})$ such that

- N is the set of players of the game;
- H is a set of sequences of actions from A, which are called *histories*, or *plays* of the game. We require that:
  (a) the empty sequence () is in H;
  (b) If $h \in H$, then any initial segment of h is in H too;
- Z is the set of maximal histories of the game;
- $P : H \setminus Z \to N$ is the player function which assignes to every nonterminal history the player whose turn is to move;
- each u_i is the payoff function for player $i \in N$, that is, a function which specifies for each maximal history what is the payoff for player i;

For any nonterminal history $h \in H$ we define

$$A(h) = \{x \in A : h \frown x \in H\}$$

A strategy for a player i is any function

$$f_i : P^{-1}(\{x\}) \to A$$

such that $f_i(h) \in A(h)$, where $P^{-1}(\{x\}$ is the set of all histories where player i is to move.

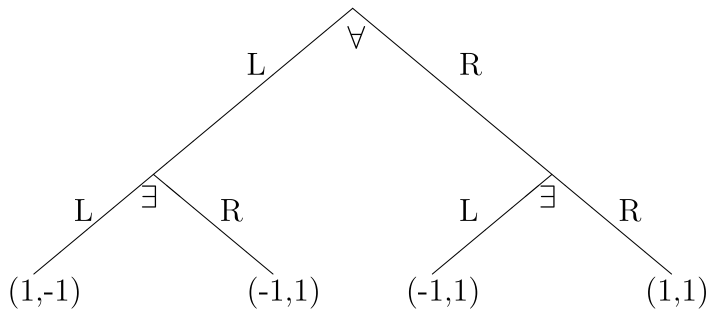
From the class of extensive games of perfect information, we single out a particular subclass, which is the class of z*ero-sum* (*win-loss*) games. These are games played by two players, that is, $N = \{\exists \text{ (Eloise)}, \forall \text{ (Abelard)}\}$, and in addition:

- $u_\exists(h) = -u_\forall(h)$ (the game is competitive), for all terminal histories h.
- $u_\exists(h) = 1$ or $u_\exists(h) = -1$ (that is, $\exists$ either wins or loses), for all terminal histories $h \in H$
- $u_\forall(h) = 1$ or $u_\forall(h) = -1$, for all terminal histories $h \in H$.

The following theorem is an old result due to Zermelo:

Every finite extensive zero-sum game is determined: either player $\exists$ or player $\forall$ has a winning strategy in the game.

The password game of perfect information: *Abelard* tells *Eloise* a password R or L. If *Eloise* is able to repeat it later on, she wins, and *Abelard* loses. Otherwise *Abelard* wins and *Eloise* loses.

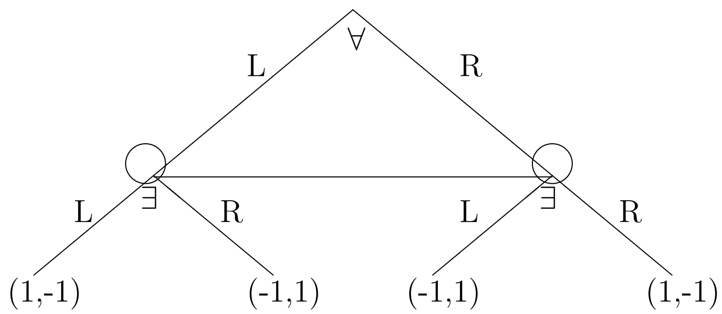


$Figure 1$

Player's $\exists$ winning strategy is straightforward: $f_\exists(L) = L, f_\exists(R) = R$.

2 Extensive games of imperfect information: general case

The games are exactly as before, except that the players might not know what happened earlier in the game. A different way to say the same thing is that players might not distinguish between histories of the game. Consider the example of the password game above, except that now player ∃ does not know (or forgets) the password given to him by player ∀. Then in the extensive form of the game, he would not distinguish between the histories L and R. In other words, the two histories are equivalent for player ∃, a fact marked in the picture by a line:



$Figure 2$

Actually every single history is indistinguishable from itself, so in the picture we also have indicated (by circles) the equivalence of the relevant histories to themselves. We then require that the strategy function f be uniform on equivalent histories, that is,

$$f(h) = f(h'), for any equivalent histories h, h' \in H$$

Imperfect information does mainly three things:

- It introduces an equivalence relation E on the class of histories of the game.
- It introduces indeterminacy in the game.
- It allows for a phenomenon known in game theory as *signalling*.

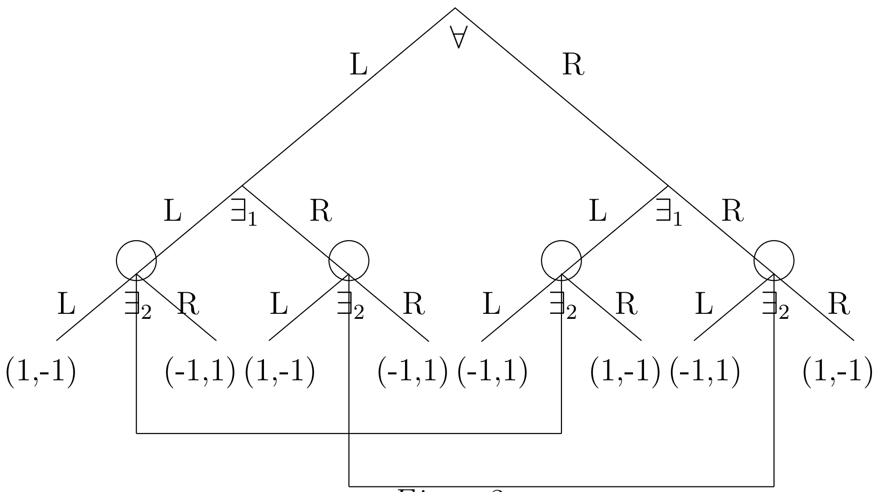
The requirement of the uniformity of the strategy functions of the players can now be expressed by:

$$E(h,k) \Rightarrow f_i(h) = f_i(k), for i \in \{\exists, \forall\}.$$

The password game in the picture above provides an example of indeterminacy: neither player ∃ nor player ∀ has a winning strategy in the game.

An example of signalling is provided by a modification of the password game.

The extended password game. There are two teams: the team of *Abelard* (consisting of *Abelard* only) is playing against the team of two players $\exists_1$ and $\exists_2$. The game is played in the following way: *Abelard* tells player $\exists_1$ a password L or R (without player $\exists_2$ hearing it), after which player $\exists_1$ tells a password L or R to player $\exists_2$. Finally, if player $\exists_2$ is able to repeat the password told by *Abelard*, the team $\exists_1$ and $\exists_2$ wins; otherwise ∀ wins.



$Figure 3$

The equivalence relation of the game is:

$$E = \{((L, L), (R, L)), ((R, R), (L, R))\} \cup \{(h, h) : h \in H\}.$$

Notice what is going on here: Although player $\exists_2$ does not "see" the choice made by player ∀, player $\exists_1$ can reveal to her what this choice was. This will provide the team $\{\exists_1, \exists_2\}$ with a winning strategy, consisting of two functions

$$f_{\exists_1} : \{L, R\} \to \{L, R\}, f_{\exists_2} : \{L, F\} \times \{L, R\} \to \{L, R\}$$

such that

$$f_{\exists_1}(L) = L, f_{\exists_1}(R) = R$$

$$f_{\exists_2}(L,L) = f_{\exists_2}(R,L) = L,\ f_{\exists_2}(L,R) = f_{\exists_2}(R,R) = R$$

Notice that, for any $S \in \{L, R\}$:

$$u_{\exists_1}(S, f_{\exists_1}(S), f_{\exists_2}(S, f_{\exists_1}(S))) = u_{\exists_2}(S, f_{\exists_1}(S), f_{\exists_2}(S, f_{\exists_1}(S))) = 1.$$

3 Semantical games of perfect information

An extensive semantical game $\mathcal{G}(\varphi, M)$ of perfect information associated with a formula φ and a model M is exactly like a zero-sum extensive game defined above, except that it has one extra-element: a labelling function $L : H \to Subform(\varphi)$ such that

- $L(()) = \varphi$
- For every terminal history h : $L(h)$ is an atomic formula of φ or its negation.

In addition, H, L, P, $u_\exists$ and $u_\forall$ satisfy jointly the following requirements:

- If $L(h) = \psi \vee \theta$ ($\psi \wedge \theta$), then $h \frown Left \in H$, $h \frown Right \in H$, $L(h \frown Left) = \psi$, and $L(h \frown Right) = \theta$.
- If $L(h) = \psi \vee \theta$ ($\psi \wedge \theta$), then $P(h) = \exists$ ($\forall$) (we assume that φ is in negation normal form).
- For every terminal history h :

  (a) If $L(h) = p$ and $M \models p$, then $u_\exists(h) = 1$ and $u_\forall(h) = -1$.

  (b) If $L(h) = p$ and not $M \models p$, then $u_\exists(h) = -1$ and $u_\forall(h) = 1$.

The above rules say in a precise way what could be describe informaly as: a disjunction (conjunction) prompts a move by player $\exists$ ($\forall$) who chooses $Left$ or $Right$. The game stops with an atomic sentence or its negation. If that sentence is true, then $\exists$ wins; otherwise $\forall$ wins.

The notion of strategy is defined in the same way as above. In addition we will define ('$M \models^+_{GS} \varphi$' means 'φ is true in M in the game semantics' and '$M \models^-_{GS} \varphi$' means 'φ is false in M in the game semantics'):

(a) $M \models^{+}_{GS} \varphi$ if and only if there is a winning strategy for player $\exists$ in the game $\mathcal{G}(\varphi, M)$.

(b) $M \models^{-}_{GS} \varphi$ if and only if there is a winning strategy for player $\forall$ in the game $\mathcal{G}(\varphi, M)$.

The extensions of the above definitions for quantifiers are straightforward. We add to the aforementioned rules the following:

- If $L(h) = \exists x\varphi$ $(\forall x\varphi)$, then $h \frown a \in H$, for every $a \in Dom(M)$.
- If $L(h) = \exists x\varphi$ $(\forall x\varphi)$, then $P(h) = \exists$ $(\forall)$ (we assume that φ is in negation normal form).
- For every terminal history h :

  (a) If $L(h) = Rt_1, ..., t_n$ and $M \models Rt_1, ..., t_n[h]$, then $u_{\exists}(h) = 1$ and $u_{\forall}(h) = -1$.

  (b) If $L(h) = Rt_1, ..., t_n$ and not $M \models Rt_1, ..., t_n[h]$, then $u_{\exists}(h) = -1$ and $u_{\forall}(h) = 1$.

The extensive semantical game $\mathcal{G}((p \vee q) \wedge (q \vee p)), M)$, with $M = \{p\}$ is given in the picture below:

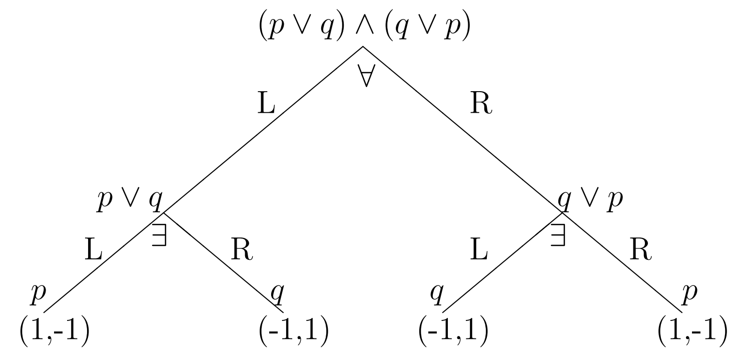


Figure4

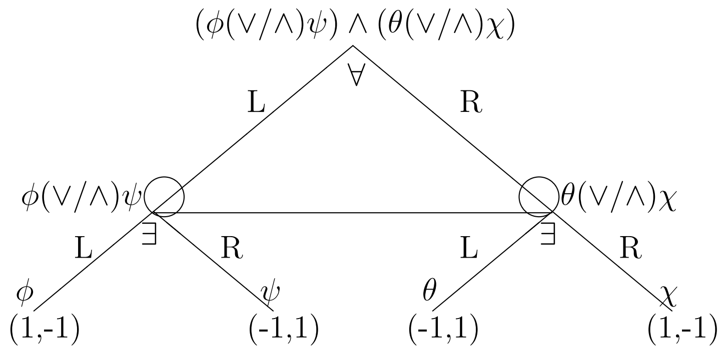
The winning strategy of player $\exists$ is: $f_{\exists}(L) = L, f_{\exists}(R) = R$.

4 Extensive semantical games of imperfect information

These are exactly like extensive semantical games of perfect information, except that now the players will not be able to distinguish between histories of the game. The histories which are indistinguishable are united by a line, as usual in such a case. However, this device of indicating the equivalence of the relevant histories is dispensable, because it can be recovered from the formulas which label the histories of the tree. That is, in the case of connectives, we shall have labelling formulas of the form

$$(\varphi\ (\vee/\wedge)\ \psi) \wedge (\theta\ (\vee/\wedge)\ \chi)$$

(to be red: 'the choice prompted by the disjunction sign is informationally independent by that prompted by the conjunction sign') where the slash indicates that the player $\exists$ cannot distinguish the histories $L(eft)$ and $R(ight)$ chosen earlier in the game by player $\forall$ in connection with the conjunction sign. In the picture below we have the extensive form of such a game. We use both the slash and the lines to indicate the lack of information:



$Figure 5$

As in the general case of games of imperfect information, the strategy functions of the players are required to be uniform, i.e. constant for equivalent (indistinguishable) histories. Thus, assuming that φ, ψ, θ, and χ are atomic propositional formulas, we have

$$M \models^+_{GS} (\varphi\,(\vee/\wedge)\,\psi)\wedge(\theta\,(\vee/\wedge)\,\chi) \Leftrightarrow (M \models^+ \varphi and M \models^+ \theta) or (M \models^+ \psi and M \models^+ \chi)$$

$$M \models^-_{GS} (\varphi\,(\vee/\wedge)\,\psi)\wedge(\theta\,(\vee/\wedge)\,\chi) \Leftrightarrow (M \models^- \varphi and M \models^- \psi) or (M \models^- \theta and M \models^- \chi)$$

However, in a model M in which φ and χ are true, but ψ and θ are false, none of the two players has a winning strategy. Thus we regain here the same indeterminacy of games that we found in the game in Figure 2.

The same phenomena occur at the level of quantifiers. In this case we shall have quantifiers of the form $(\exists x/W)$ where W is a (possibly empty) set of variables. The idea here is very simple: the choice prompted by $\exists x$ is informationally independent of choices prompted by quantifiers bounding the variables in the set W.

We give below two examples. The first one is an indeterminate semantical game. The second one is a variation of the first, which is determinate and which illustrates the device of signalling.

Here is the extensive form of the game $\mathcal{G}(\forall x_0(\exists x_1/\{x_0\}x_0 = x_1, M)$, where $M = \{L, R\}$:

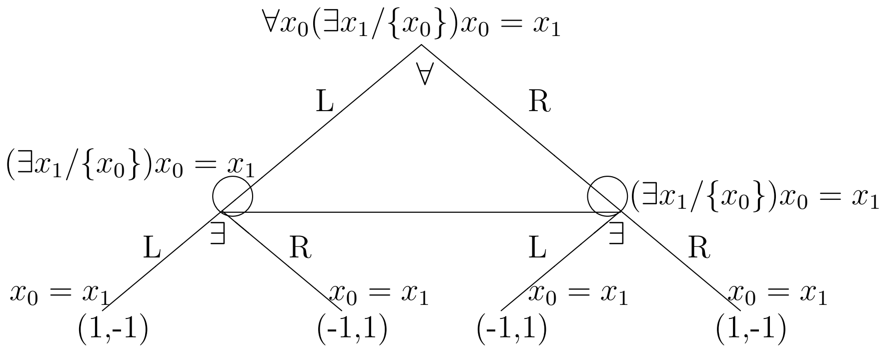


*Figure*6

The equivalence relation of the game is

$$E = \{(L, R), (R, L)\} \cup \{(h, h) : h \in H\}.$$

Notice that there is no uniform function $f : \{L, R\} \rightarrow \{L, F\}$ such that for all $S \in \{L, R\} : S = f(S)$.

Similarly, there is no $S \in \{L, R\}$ such that for all $P \in \{L, R\} : S = P$.
Thus neither $\exists$ nor $\forall$ has a winning strategy in the game.

An interesting variation of the previous game is formed by adding a dummy variable to the sentence $\forall x_0(\exists x_1/\{x_0\}x_0 = x_1$. That is, we get the game $\mathcal{G}(\forall x_0 \exists x_1(\exists x_2/\{x_0\}x_0 = x_2, M), M = \{L, R\}$, whose extensive form is given below:

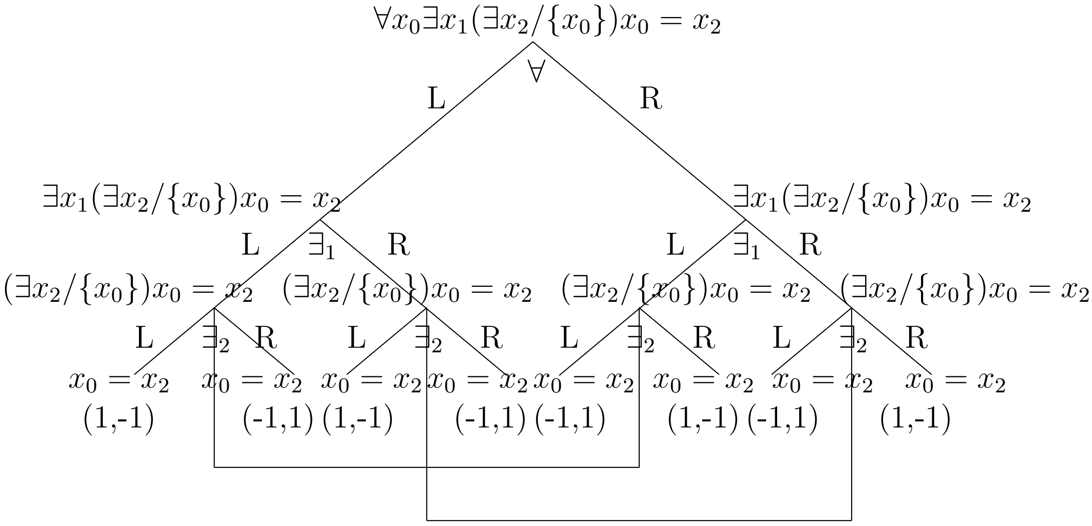


*Figure*7

The equivalence relation of the game is:

$$E = \{((L, L), (R, L)), ((R, R), (L, R))\} \cup \{(h, h) : h \in H\}.$$

The winning strategy of player $\exists$ consists of two functions exactly as in the extended password game of imperfect information from Example 2.1.

5 Digression: an extension of propositional logic

Let the usual propositional language $L(\sigma)$ (in the signature σ) extended with a four-place connective $W(\varphi, \psi, \theta, \chi)$ be $L(W)(\sigma)$, that is, the smallest set of sentences closed with respect to the sentences in σ, the familiar rules for the connectives in the set $\{\neg, \vee\}$, and the following rule for the new connective W:

- If φ, ψ, θ, and χ are $L(W)(\sigma)$-sentences, then so is $W(\varphi, \psi, \theta, \chi)$.

The semantics involves the notions $M \models^+ \varphi$ (the sentence φ is true in M) and $M \models^- \varphi$ (the sentence φ is false in M), and is defined by a double induction on the length of φ.

(i) $M \models^+ S \Leftrightarrow S \in M$

(ii) $M \models^- S \Leftrightarrow S \notin M$

(iii) $M \models^+ (\neg\varphi) \Leftrightarrow M \models^- \varphi$

(iv) $M \models^- (\neg\varphi) \Leftrightarrow M \models^+ \varphi$

(v) $M \models^+ (\varphi \vee \psi) \Leftrightarrow M \models^+ \varphi$ or $M \models^+ \psi$

(vi) $M \models^- (\varphi \vee \psi) \Leftrightarrow M \models^- \varphi$ and $M \models^- \psi$

(vii) $M \models^+ W(\varphi, \psi, \theta, \chi) \Leftrightarrow (M \models^+ \varphi$ and $M \models^+ \theta)$ or $(M \models^+ \psi$ and $M \models^+ \chi)$

(viii) $M \models^- W(\varphi, \psi, \theta, \chi) \Leftrightarrow (M \models^- \varphi$ and $M \models^- \psi)$ or $(M \models^- \theta$ and $M \models^- \chi)$.

Thus the new connective $W(\varphi, \psi, \theta, \chi)$ is the one extracted from our games of imperfect information in the previous section.

In classical logic L, every truth-function $f\colon \{0,1\}^n \to \{0,1\}$ from a class of models K into $\{0,1\}$ is definable by a sentence in L such that for all $M \in \mathsf{K}$, $\|\varphi\|^M = f(M)$. This result extends to $L(W)$.

(Sandu & Pietarinen) Let K be a class of classical models in σ. For any function f from K into $\{0, 1, ?\}$, there is an $L(W)(\sigma)$-sentence φ such that $\|\varphi\|^M = f(M)$ for all $M \in \mathsf{K}$.

The interpretation of negation $\neg$ makes it strong, positive negation, transforming truths to falsehoods and falsehoods to truths, but not meddling with non-determinable values. There is a version of weak, contradictory negation available, defined in the following way. Let $L(W, \neg_w)$ be $L(W)$ extended with a contradictory negation $\neg_w$. Definition 5.1 is augmented with two clauses as follows.

For any $L(W, \neg_w)(\sigma)$-sentence φ and a model M:

(i) $M \models^+ (\neg_w\varphi)$ iff not $M \models^+ \varphi$

(ii) $M \models^- (\neg_w\varphi)$ iff not $M \models^- \varphi$.

If $M = \{S_1, S_4\}\rangle$, then we have neither $M \models^+ W(S_1, S_2, S_3, S_4)$ nor $M \models^- W(S_1, S_2, S_3, S_4)$. But then by the above Definition, we have both $M \models^+ (\neg_w W(S_1, S_2, S_3, S_4))$ and $M \models^- (\neg_w W(S_1, S_2, S_3, S_4))$.

Consequently, one sees that the presence of weak negation introduces a fourth truth-value. Therefore, the interpretation of an $L(W, \neg_w)(\sigma)$-sentence φ can have the following values.

The next question is whether $L(W, \neg_w)$ has a functionally complete set of connectives. The answer is affirmative, as shown by the next theorem.

(Sandu & Pietarinen) Let K be a class of classical models in σ. For any function $f\colon \mathsf{K} \to \{0, 1, ?, !\}$, there is an $L(W, \neg_w)(\sigma)$-sentence φ such that $\|\varphi\|^M = f(M)$ for all $M \in \mathsf{K}$.

6 Modelling extensive games by semantical games

In simple cases, extensive zero-sum games $\mathcal{G}_A = (\{\exists, \forall\}, H, Z, P, (u_i)_{i\in\{\exists,\forall\}})$ of perfect information can be modelled by extensive semantical games $\mathcal{G}(\varphi, M) = (\{\exists, \forall\}, H, Z, P, L, (u_i)_{i\in\{\exists,\forall\}})$ of perfect information in a straighforward way. By "modelling" we mean that the formula φ and the structure M of the semantical game $\mathcal{G}(\varphi, M)$ are completely determined by $\mathcal{G}_A$, and in addition we have:

F (G) is a set of winning strategy functions for player $\exists$ $(\forall)$ in $\mathcal{G}_A$ iff
F (G) is a set of winning strategy functions for player $\exists$ $(\forall)$ in $\mathcal{G}(\varphi, M)$.

By a "simple case" we mean one in which all the maximal histories of the game $\mathcal{G}_A$ have the same length, and all the non maximal histories of the same length are labelled by the same player, that is, $P(h)$ is constant for all nonmaximal histories $h \in H$ of the same length. The first-order sentence φ to be extracted from such a game is the following sentence in prenex normal form

$$Q_0 x_1 ... Q_{n-1} x_{n-1} R x_0 ... x_{n-1}$$

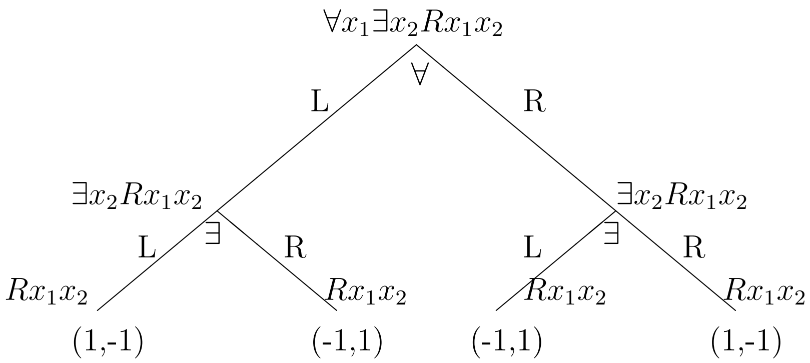
where each Q_i is either $\forall$ or $\exists$,and in adition

$$Q_i = P(h), h \in H, length(h) = i$$

The model $M = (\mid M \mid, R^M)$ is then defined as:

$$\mid M \mid = A,\ R^M = \{h \in H : u_\exists(h) = 1\}.$$

Recall the password game of perfect information from the first section (cf. Figure 1). The semantical game corresponding to it is $\mathcal{G}(\forall x_1 \exists x_2 Rx_1x_2, M)$, where $\mid M \mid = \{L, R\}$, $R^M = \{(L, L), (R, R)\}$, whose extensive form is given in the picture below:



$Figure 8$

The correspondence established above can be extended to extensive games of imperfect information.

Let $\mathcal{G}_A = (\{\exists, \forall\}, H, Z, P, (u_i)_{i \in \{\exists, \forall\}})$ be a zero-sum game of imperfect information in which all the maximal histories have the same length, and all the histories of the same length are labelled by the same player, that is, $P(h)$ is constant for all histories $h \in H$ of the same length. For every $m, m \geq 0, m \leq n-1$, let H^m be the class of all histories of length m. Let W_m be a (possibly empty) set of natural numbers, $W_m \subseteq m = \{0, ..., m-1\}$. W_m determines an equivalence relation E^m_W on H^m defined in the following way. Let $h, k \in H^m$, $h = (h_0, ..., h_{m-1})$, $k = (k_0, ..., k_{m-1})$. Then

$$(h, k) \in E^m_W \Leftrightarrow \forall i \in (m - W_m) : h_i = k_i$$

Notice that in case $W_m = \emptyset$, we just have the standard equivalence relation:

$$E^n_\emptyset = \{(h, h) : h \in H^m\}.$$

We say that a zero-sum game $\mathcal{G}_A = (\{\exists, \forall\}, H, Z, P, (u_i)_{i \in \{\exists, \forall\}})$ is a *uniform* game of imperfect information if all the above conditions are satisfied, i.e.

- All the maximal histories have the same length;

- For every $m \geq 0$, $P(h)$ is constant for all $h \in H^m$;
- The equivalence relation E on the histories of $\mathcal{G}_A$ can be written in the form:

$$E = \bigcup \{E^m_W : E^m_W is an equivalence relation on H^m for some W_m \subseteq m\}.$$

It can be easily proved that every uniform zero-sum game $\mathcal{G}_A = (\{\exists, \forall\}, H, Z, P, (u_i)_{i \in \{\exists,\forall\}})$ of imperfect information can be modelled by an extensive semantical game of imperfect information $\mathcal{G}(\varphi, M) = (\{\exists, \forall\}, H, Z, P, L, (u_i)_{i \in \{\exists,\forall\}})$, where φ is an $IF-$ sentence in prenex normal form

$$(Q_0 x_0/W_0)...(Q_{n-1}x_{n-1}/W_{n-1})Rx_1...x_{n-1}$$

such that

- Each Q_i is either $\forall$ or $\exists$,
- $Q_i = P(h)$, $h \in H$, $length(h) = i$,
- Each (possibly empty) W_i is defined as: $W_i = \{x_j : j \in W, E^i_W$ is the equivalence relation on $H^i\}$
- The model $M = (\mid M \mid, R^M)$ is determined in the same way as in the case of perfect information.

Recall the password game of imperfect information from the second section (cf. Figure 2). It certainly satisfies all the requirements discussed above. We have two equivalence relations:
$E^0_\emptyset = \{((), ())\}$
$E^1_{\{0\}} = \{(L, R), (R, L), (L, L), (R, R)\}$.
Thus the corresponding semantical game is $\mathcal{G}(\forall x_0(\exists x_1/\{x_0\})Rx_0x_1, M)$, which was described in Example 4.1.

Recall the extended password game of section 2 (Figure 3). The corresponding equivalence relations are:
$E^0_\emptyset = \{((), ())\}$;
$E^1_\emptyset = \{((L), (L)), ((R), (R))\}$;
$E^2_{\{0\}} = \{((L, L), (R, L)), ((L, R), (R, R)), ((L, L), (L, L)), ((R, R), (R, R))\}$
Thus the corresponding semantical game is $\mathcal{G}(\varphi, M)$ where

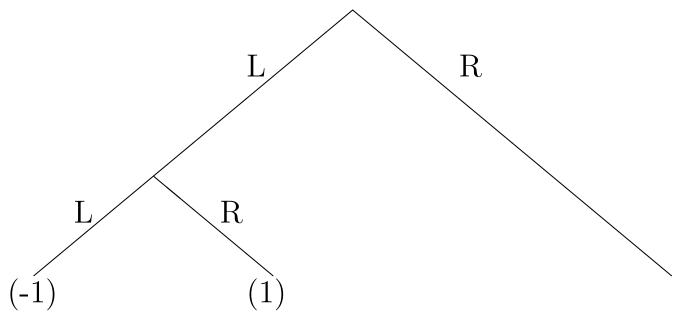
$$\varphi : \forall x_0 \exists x_1 (\exists x_2/\{x_0\}) Rx_0x_1x_2$$

$$M = (\{L, R\}, \{(L, L, L), (L, R, L), (R, R, R)\}, (R, L, R)\})$$

whose extensive form is exactly like that of the game in Figure 7, except that $Rx_0x_1x_2$ replaces $x_0 = x_2$.

There is another way to represent the extended password game by a semantical game of imperfect information. If we look at the way the payoffs are determined, we see that what matters is the identity of the first and choice set in the game. The second choice matters only for the "signalling" of the strategy of player $\forall$ to player $\exists_2$ but this choice itself is not operative in determining the payoffs. Thus another equally faithful representation of the password game will be by the semantical game $\mathcal{G}(\forall x_0 \exists x_1 (\exists x_2/\{x_0\}) x_0 = x_2, M)$, with $M = \{L, R\}$. This is exactly the game from Figure 7.

Admittedly, the restrictions we put on the extensive games of imperfect information are bound to leave out plenty of interesting cases. One such a case is the game of the absent minded driver (Rubinstein, 1988). A driver, in order to get home has to take the highway, but then he has two possibilities: either to get out at the first exit or at the second one. Both of them will take him home but he does not remember which is which. That is, when he is at the intersection he does not remember if he has been there before or not. In order to make things simpler, we may suppose that the pay off he gets in both cases is the same, that is 1, if he takes the exit, and -1 if he goes to the highway.



$Figure 9$

Notice that in this game, where L represents the choice of going to the highway and R of taking the exit, we have to equivalent histories of different length: () and L.

The issue of the equivalence of histories of different length is discussed in van Benthem (forthcoming).

It is also interesting to notice that both the finite extensive games of perfect information and of imperfect information can be modelled by extensive semantical games $\mathcal{G}(\varphi, M)$ where φ is a propositional $IF-$formula. Without going for a more general characterization of such an embedding, we suffice ourselves to give here two examples corresponding to the two versions of the password game.

The simpler password game of the second section may be represented by the semantical game $\mathcal{G}((p\ (\vee/\wedge)\ q) \wedge (q\ (\vee/\wedge)\ p), M)$, with $M = \{p\}$. The extensive form of this game is the same as the one given earlier for $(\varphi\ (\vee/\wedge)\ \psi) \wedge (\theta\ (\vee/\wedge)\ \chi)$.

The extended password game may be represented by the semantical game $\mathcal{G}([(p\ (\vee/\wedge)\ q) \vee (q\ (\vee/\wedge)\ p)] \wedge [(q\ (\vee/\wedge)\ p) \vee (p\ (\vee/\wedge)\ q)], M)$, with $M = \{p\}$. The extensive form of this game should be obvious.

7 Semantical games as Kripke structures

It turns out to be interesting to compare informationally independent connectives with informationally independent quantifiers (this is an issue elaborated in Sandu & Pietarinen, forthcoming). For this purpose, let us return to our earlier example $(\varphi\ (\vee/\wedge)\ \psi) \wedge (\theta\ (\vee/\wedge)\ \chi)$. Hodges (1997) pointed out that this sort of informational independence does not make much sense in the game-theoretical setting, for after $\forall$ has made the first move choosing one of the conjuncts, then it is $\exists$'s turn to move. But when she moves, it is reasonable to require that she knows the set of her possible choices: if she is supposed to choose from the set $\{\varphi, \psi\}$ then she can realize or infer that previous to her move $\forall$ has chosen the left conjunct. If her choice is from the set $\{\theta, \chi\}$, then she may infer that $\forall$ has chosen the right conjunct. In both case, the informational independence of $\exists$'s move from that of $\forall$ is cancelled!

We think that the problem raised by Hodges is a deep one. However, in the preceding section, we were able to circumvent it by introducing a split between, on one side, the choices made by the players (i.e. the actions of the players), and on the other, the formula with which the game goes on (i.e. the labelling formulas of the game.). In the rules for conjunction and disjunction, the players did not have to choose between the left and the right *disjunct* (*conjunct*), but simply between Left or Right, and the game would go on with the formula which is the left or the right disjunct (conjunct).

In order to spell out the difference between Hodges' approach and the present one, it may be useful to look at games as Kripke structures. The conversion of an extensive zero-sum semantical game $\mathcal{G}(\varphi, M) = (\{\exists, \forall\}, H, Z, P, L, (u_i)_{i \in \{\exists,\forall\}})$ into a Kripke structure $K_{\mathcal{G}} = (W, R_\exists, R_\forall, V)$ is straightforward (φ is a propositional formula):

- $W = H$
- $R_\exists = \bigcup\{E^n_W : n$ is the length of a history $h \in H$ such that $P(h) = \exists\}$
- $R_\forall = \bigcup\{E^n_W : n$ is the length of a history $h \in H$ such that $P(h) = \forall\}$
- $V(h, p) = t \Leftrightarrow h$ is a maximal history and $u_\exists(h) = 1$, for p an atomic sentence.

We shall now introduce more operators in our language.

Let **Choice** be an operator on the actions of the players so that '$K_{\mathcal{G}}, h \models$ **Choice**(a)' is read 'the action a is an option for the player who is to move at the history h'. The interpretation is straighforward:

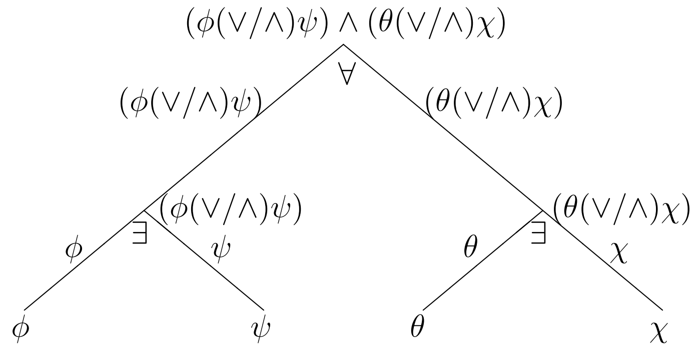
$$K_{\mathcal{G}}, h \models \textbf{Choice}(a) if and only if h \frown (a) \in H$$

(We should have names in the object language for the actions of the players; however, we prefer to do without, since no confusion arises.)

We introduce another operator **Label** in the object language corresponding to the labelling function $L : H \rightarrow Subform(\varphi)$. We then define the notion '$K_{\mathcal{G}}, h \models$ **Label**(φ)' (the history h is labelled with the formula φ):

$$K_{\mathcal{G}}, h \models \textbf{Label}(\varphi) iff L(h) = \varphi$$

Suppose now we do not distinguish, as we did, between the actions of the players (choosing $Left$ or $Right$) and the formulas labelling the histories of the game. That is, in the game $\mathcal{G}((\varphi\ (\vee/\wedge)\ \psi) \wedge (\theta\ (\vee/\wedge)\ \chi), M)$, the players, instead of choosing $Left$ or $Right$ choose directly subformulas of the sentence $(\varphi\ (\vee/\wedge)\ \psi) \wedge (\theta\ (\vee/\wedge)\ \chi)$. The extensive form of the game will be slightly modified to take into account this fact:



$Figure 10$

In virtue of our earlier definitions we have:

$$K_{\mathcal{G}}, (\varphi\ (\vee/\wedge)\ \psi) \models \neg K_{\exists}\mathbf{Choice}(\varphi) \wedge \neg K_{\exists}\mathbf{Choice}(\psi)$$

$$K_{\mathcal{G}}, (\theta\ (\vee/\wedge)\ \chi) \models \neg K_{\exists}\mathbf{Choice}(\theta) \wedge \neg K_{\exists}\mathbf{Choice}(\chi)$$

On the other side, in the approach we have adopted, we have (cf. Figure 5):

$$K_{\mathcal{G}}, L \models K_{\exists}\mathbf{Choice}(L) \wedge K_{\exists}\mathbf{Choice}(R)$$

$$K_{\mathcal{G}}, R \models K_{\exists}\mathbf{Choice}(L) \wedge K_{\exists}\mathbf{Choice}(R)$$

So Hodges is right: if we want the players to have knowledge of their choices, then we have to give up the assumption of informational independence (which tecnically speaking amounts to giving up the equivalence between the histories ($\varphi\ (\vee/\wedge)\ \psi$) and ($\theta\ (\vee/\wedge)\ \chi$)). But notice, once again, that this holds only for the case in which the actions of the plyers are not distinguished from the labelling formulas of the game tree.

In the approach we have favored in this paper, we endowed the players with knowledge of the choices they have at each history. On the other side, the present approach is not without its problems, because it can be seen that the players are forced to choose 'blindly': unlike in Hodges' approach, the player know, at each history, that they have to choose between L and R, but they do not know, when it is their turn to move, which is the formula which labels the node at which they are. That is, the following are seen to hold in the setting of this paper (cf. Figure 5):

$$K_{\mathcal{G}}, L \models K_{\exists}[\mathbf{Label}((\varphi\ (\vee/\wedge)\ \psi)) \vee \mathbf{Label}((\theta\ (\vee/\wedge)\ \chi))]$$

and

$$K_{\mathcal{G}}, R \models K_{\exists}[\mathbf{Label}((\varphi\ (\vee/\wedge)\ \psi)) \vee \mathbf{Label}((\theta\ (\vee/\wedge)\ \chi))]$$

but on the other side

$$K_{\mathcal{G}}, L \models \neg K_{\exists}\mathbf{Label}((\varphi\ (\vee/\wedge)\ \psi)) \wedge \neg K_{\exists}\mathbf{Label}((\theta\ (\vee/\wedge)\ \chi))$$

$$K_{\mathcal{G}}, R \models \neg K_{\exists}\mathbf{Label}((\varphi\ (\vee/\wedge)\ \psi)) \wedge \neg K_{\exists}\mathbf{Label}((\theta\ (\vee/\wedge)\ \chi)).$$

None of these problems arise in the case of independent quantifiers. That is, in this case the players have full knowledge of both their actions and the labelling formulas which are the result of these actions. Consider, as an example, our earlier game $\mathcal{G}(\forall x_0(\exists x_1/\{x_0\})Rx_0x_1, M)$, with $M = \{L, R\}$ (cf. Figure 6):

Then in the corresponding Kripke structure $K_{\mathcal{G}}$, we have:

$$K_{\mathcal{G}}, L \models K_{\exists}\mathbf{Label}(\exists x_1/\{x_0\})Rx_0x_1)$$

$$K_{\mathcal{G}}, R \models K_{\exists}\mathbf{Label}(\exists x_1/\{x_0\})Rx_0x_1)$$

and also

$$K_{\mathcal{G}}, L \models K_{\exists}\mathbf{Choice}(L) \wedge K_{\exists}\mathbf{Choice}(R)$$

$$K_{\mathcal{G}}, L \models K_{\exists}\mathbf{Choice}(L) \wedge K_{\exists}\mathbf{Choice}(R)$$

So in view of the fact that in the case of informationally independent quantifiers, the players have full knowledge of the course of the game, we should perhaps follow the recommendation given by Hodges (1997) and treat connectives as restricted quantifiers. In this case the formula $(\varphi\ (\vee/\wedge)\ \psi)$ and $(\theta\ (\vee/\wedge)\ \chi))$ will have to be rewritten as $\forall i_1(\exists i_2/\{i_1\})\, p_{i_1 i_2}$, where i_1 and i_2 range over the domain $\{1, 2\}$, and in addition $p_{11} = \varphi, p_{12} = \psi, p_{21} = \theta$,and $p_{22} = \chi$.

References

Blass, A, and Y. Gurevich, 1986, 'Henkin Quantifiers and Complete Problems', Annals of Pure and Applied Logic 32, 1-16.

Hintikka, J., 1996, The Principles of Mathematics Revisited, Cambridge University Press.

Hintikka, J. and G. Sandu, 1997, 'Game-theoretical Semantics', in J. van Benthem and A. ter Meulen (eds.), Handbook of Logic and Language, Elsevier Science B.V., 361-410.

Hodges, W., 1997, 'Compositional Semantics for a Language of Imperfect Information',mathit Journal of the IPGL 5, 539-563.

Rubinstein, A., 1998, Modelling Bounded Rationality, MIT Press.

Sandu, G. and A. Pietarinen, (forthcoming in the Journal of IPGL), 'Partiality and Games: Propositional Logic.

Sandu, G. and J. Väänänen, (1992), 'Partially Ordered Connectives', Z. Math. Logik Grundl. Math. 38, 361-372.

Van Benthem, Johan, (forthcoming), 'Hintikka Self-applied' (manuscript).