**Jan Woleński**

# NOTES ON METALOGIC

# ERRATA

p. 10, Remark 4, l. 11 (bottom), add "has" before "additionally".

p. 10, Remark 4, l. 9 (bottom), should be "transform (**A1**) – (**A15**) into concrete formulas as well as (**A'1**) – (**A'3**) into schemata.

p. 11, Remark 5, l. 4 (top), cancel "provided that".

p. 12, l. 3 (bottom), cancel "If $B = A$, then $(A \rightarrow B) = (A \rightarrow A)$".

p. 12, l. 3 (bottom), replace ".if" by ".If".

p. 13, l. 13 (bottom), replace "(b) that observe" by "(b) observe that".

p. 15, (**C1**), replace $\geq$ by $\leq$.

p. 22, l. 9 (bottom), replace $\mathbf{V}(A) = \mathbf{0}$ by $\mathbf{V}(A) = \mathbf{1}$.

p. 26, l. 7 (bottom), cancel the first occurence of "($n > 1$)".

p. 28, l. 9 (top), insert "not" after "are".

p. 29, **(D24)**, replace "for every *X*" by "for every *A*".

p. 31, **(GMPL)**, insert "theory" after "first order".

p. 31, l. 15 (bottom), add (b) $\Phi(c_i) = c_i$.

p. 31, **(COPL)**, replace "first-formulas" by "first-order formulas".

# ESSLLI – 2001

**Jan Woleński**

# NOTES ON METALOGIC

## §1. Introduction.

Alexander of Aphrodisias, an Aristotelian scholar ordered the works of the Stagirite in a way. In Alexander's catalogue, Aristotle's work devoted to the first philosophy (*prote filosofia*) was placed just after the book *Physics*. Thus, the word "metaphysics" arose as a composition of "meta" (after) and "physics" and originally meant "after physics". However, some historians of ancient philosophy suggest that our word was intentionally introduces in order to point out considerations of a special kind, namely reflection about nature (*physis*) and its theory. Anyway, this more substantial application of the word "metaphysics" very soon became official. Today, metaphysics is considered as the theory of being and very often identified with ontology.

The use of words beginning with the prefix "meta" began quite popular in 20th century. One can mention "metatheory", "metascience", "metaethics", "meta-mathematics" or "metalogic" as examples. Their intended meaning consists in pointing out some considerations about fields indicated after the prefix "meta". The word "metaphysics" would be a good label for methodology of physics, but it is excluded but its history mentioned above. The word "metatheory" denotes or perhaps suggests a theory of theories. Metascientific studies in the 20th century used the term "metatheory" to refer to investigations of theories in a variety of disciplines, for example, logic, sociology, psychology, history, etc. However, the philosophers of the Vienna Circle, who made metatheoretical studies of science the main concern of their philosophy, restricted metatheory to the logic of science modelled on developments in the foundations of mathematics. More specifically, the logic of science was intended to play a role similar to metamathematics in Hilberts's sense; that is, it was projected as formal analysis of scientific theories understood as well-defined linguistic items. The word "metamathematics" was also used before Hilbert, but with a different meaning from his. In the early 19th century, mathematicians, like Gauss, spoke about metamathematics in an explicitly pejorative sense. It was for them a speculative way of looking at mathematics - a sort of metaphysics of mathematics. A negative attitude to metaphysics was at that time inherited from Kant and early positivists. The only one serious use of "metamathematics" was restricted to metageometry, and that was due to the fact that the invention of different geometries in the $19^{th}$ century stimulated comparative studies. For example, investigations were undertaken of particular axiomatizations, their mutual relations, models of various geometrical systems, and attempts to prove their consistency. The prefix "meta" presently suggests two things. Firstly, it indicates that metatheoretical considerations appear "after" (in the genetic sense) theories are formulated. Secondly, the prefix "meta" suggests that every metatheory is "above" a theory which is the subject of its investigations. It is important to see that "above" does not function as an evaluation but only indicates the fact that metatheories operate on another level than theories do. A simple mark of this fact consists in the fact that theories are formulated in an object language, and metatheories are expressed in a related metalanguage.

Since metalogic is a part of metamathematics, it is convenient to say few words about the latter. It is probably not accidental that Hilbert passed to metamathematics through his famous study of geometry and its foundations. Hilbert projected metamathematics as a rigorous study of mathematical theories by mathematical methods. Moreover, the Hilbertian metamathematics, due to his views in the philosophy of mathematics (formalism) was restricted to finitary methods. If we reject this limitation, metamathematics can be described as study of mathematical systems by arbitrary mathematical methods; they cover those that are admitted in ordinary mathematics, including infinitistic or infinitary; the latter use for instance, the axiom of choice or transfinite induction. However, this description is still too narrow. Hilbert's position in metamathematics can be described as follows: only syntactic or combinatorial methods are admissible in metatheoretical studies. When the Hilbertians proved theorems with semantic content about formal systems, they used semantic concepts, like that of validity or truth rather in informal understanding. Due to works of Alfred Tarski, semantics became a rigorous mathematical field and entered the domain of metamathematics. It is perhaps interesting that the borderline between syntax and semantics corresponds to some extent to the frontier between finitary and infinitary methods. I say "to some extent" because we have also systems with infinitely long formulas (infinitary logic). It is clear that the syntax of infinitary logics must be investigated by methods going beyond finitary tools. It was also not accidental that systematic formal semantics (model theory) which requires infinitistic methods appeared in works of Alfred Tarski, who, due to the scientific ideology of the Polish mathematical school, did not accept the dogma that only finite combinatorial methods are admissible in metamathematics. Today, metamathematics can divided into three wide areas: proof theory (roughly speaking, it corresponds to metamathematics in Hilbert's sense if proof-methods are restricted to finitary tools, or it is an extension of Hilbert's position if the above-mentioned restriction is ignored), recursion theory (which is closely related to the decision problem, that is, the problem of the existence of combinatorial procedure providing a method of deciding whether a given formula is or not a theorem) and model theory, that is, studies of relations between formal systems and structures which are their realizations; model theory has many affinities with universal algebra.

Metalogic is understood here as a part of metamathematics restricted to logical systems and refers to studies of logical systems by mathematical methods. This word also appeared in the 19$^{th}$ century although its roots go back to the Midle Ages (*Metalogicus* of John of Salisbury). Philosophers, mainly Neokantians, understood metalogic to be concerned with general considerations about logic. The term "metalogic" in its modern sense was used for the first time in Poland (by Jan Łukasiewicz and Alfred Tarski) as a label for the metamathematics of the propositional calculus. Thus, metalogic is metamathematics restricted to logic, and consists of proof theory, investigations concerning the decidability problem, and model theory, all restricted to the domain of logic. When we say that metalogic is a part of metamathematics, it can suggest that the borderline between logic and mathematics can be sharply outlined. However, questions like "What is logic?" or "What is the scope of logic?" have no uniformly determined answer. We can distinguish at least three relevant subproblems that throw light on debates about the nature of logic and its scope. The first issue focuses on the so called first-order thesis. According to this standpoint, logic should be restricted to standard first-order logic. The opposite view contends that the scope of logic should be extended to a variety of other systems, including, for instance, higher-order logic or infinitary logic. The second issue focuses on the question of rivalry between various logics. The typical way of discussing the issue consists in the following question: Can we or should we replace classical logic by some other system, for instance, intuitionistic, many-valued, relevant or paraconsistent logic? This way of stating of the problem distinguishes classical logic as the system which serves as the point of reference. Thus, alternative or rival logics are identified as non-classical. There are two reasons to regard classical logics as having a special status. One reason is that classical logic appeared as the first stage in the development of logic; it is a historical and purely descriptive circumstance. The second motive is clearly evaluative in its character and consists in saying that classical logic has the most "elegant" properties or that its service for science, in particular, for mathematics, is "the best". For example, it is said that abandoning the principle of excluded middle (intutionistic logic), introducing more than two logical values (many-valued logic), changing the meaning of implication (relevant logic) or tolerating inconsistencies (paraconsistent logic) is somehow wrong. It is also often said that some non-classical logics, for example, intuitionistic or many-valued logics, considerably restrict the applicability of logic to mathematics. It is perhaps most dramatic in the case of intuitionistic logic, because it or other constructivistic logics lead to eliminating a considerable part of classical mathematics. Thus, this argument says that only classical (bivalent or two-valued) logic adequately displays the proof methods of ordinary mathematics. While the discussion is conducted in descriptive language, it appeals to intuitions and evaluations of what is good or wrong in mathematics. The situation is similar as far as the matter concerns metalogical properties of particular systems such as completeness, decidability, etc., because it is not always obvious what it means to say that a logic possesses them "more elegantly" than a rival system. The priority of classical logic is sometimes explained by pointing out that some properties of non-classical logic are provable only classically. This is particularly well-illustrated by the case of the completeness of intuitionistic logic: Is the completeness theorem for this logic intuitionistically provable? The answer is not clear, because the stock of intuitionistically or constructively admissible methods is not univocally determined, and their scope, from one author to another. Finally, our main problem (what is logic and what is its scope?) is also connected with extensions of logics. If we construct modal logics, deontic logics, epistemic logics, etc., we usually start with some basic (propositional or predicate) logic. We have modal propositional or predicate systems which are based on classical, intutionistic, many-valued or some other basic logic. Do any given extension (roughly speaking, an extension of a logic arises when we add new concepts, for example, necessity to old ones, in such a way that all theorems of the system before extension are theorems the new system) of a chosen basic logic preserves its classification as a genuine logic or do it produce an extralogical theory? The *a priori* answer is problematic, even when we decide that this or that basic system is *the* logic. The problem of the status of extensions of logic is particularly important for philosophical logic because it concerns of systems of this sort.

The three issues concerning the question "What is logic?" are mutually interconnected. The choice between first-order logic or higher-order logic automatically leads to the two other issues, because it equally arises with respect to any alternative logic and any extension of a preferred basic logic. Thus, we have a fairly complex situation. Yet the above division into three issues does not exhaust all problems. Usually it is assumed that first-order logic (classical or not) is based on the assumption that its universe it not empty. However, as Bertrand Russell once remarked, that it is a defect of logical purity, if one can infer from the picture of logic that something exists. This is perhaps the main motivation for so called free logic, that is, logic without existential assumptions (logic admitting empty domains). Is it classical or not? The described situation suggests a pessimism as far as the matter concerns a natural and purely descriptive characterization of logic; it seems that an element of a convention is unavoidable here. A further reason that the domain of metalogic cannot be sharply delimited is that several metalogical or metamathematical results distinguish logical (even in a wider sense) from other formal systems. Assume that we decide to stay with the first-order thesis. The second Gödel theorem (the unprovability of the consistency of elementary arithmetic) clearly separates pure quantification logic from formal number theory. It is one reason that metamathematical results are of interest for metalogic. Metalogical investigations also use several concepts that are defined in general metamathematics, e. g., formal system, axiomatizability, consistency, completeness, provability, etc. Fortunately, we are not forced to answer the borderline question in a final manner. These notes deal with metalogic of so classical elementary logic consisting of propositional calculus (**PC**) and first-order predicated logic (**PL**). Other systems are will be mentioned only occasionally. The treatment is rather elementary and the theorems are given without proofs in some cases. I assume some familiarity with syntax and semantics of **PC** and **PL** as s well as with several simple concepts of set theory. Special attention will be given to relations between syntactic and semantic concepts that are most strikingly displayed by the (semantic) completeness theorems. In many matters, I closely follow books by Hunter and Pogorzelski mentioned in the bibliograhy at the end of the text.

### §2. Language and syntax of PC.

Language of **PC** ($\mathbf{L_{PC}}$) is identified with the set of propositional formulas. Thus, we need to define the concept of **PC**-formula. I start with notational concentions. The letters $p_1$, $p_2$, $p_3$, ... serve as propositional variables; the symbols $\neg$ (name: negation; reading: not), $\wedge$ (name: conjunction; reading: and), $\vee$ (name: disjunction; reading: or), $\rightarrow$ (name: implication; reading: if, then), $\leftrightarrow$ (name: equivalence; reading: if and only if, iff) represent logical constants (connectives) of **PC**. Furthermore we have: the countably infinite set $\mathbf{VAR_{PC}} = \{p_1, p_2, p_3, ......\}$, the set $\mathbf{CON_{PC}} = \{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$, the set $\mathbf{AL_{PC}}$ (the alphabet **of** PC) $= \mathbf{VAR_{PC}} \cup \mathbf{CON_{PC}} \cup \{(, )\}$.

**Remark** 1. The brackets ( (left) and ) (right) will be freely used. Remember that the succession of connectives from negation to equivalence indicates the strength of particular constants with respect to brackets. Thus, the expression $\neg p_1 \vee p_2 \wedge p_2$ is to be read as $\neg(p_1) \vee (p_2 \wedge p_2)$. A pedantic style requires to use quotes and write 'the expression "E"' . However, I regard "the expression (formula, etc.) *E*" as an abbreviation for 'the expression "*E*"'. Sometimes the list of propositional variables is written as: *p*, *q*, *r*,..... It is improper because it suggests that the symbols for propositional variables are restricted only to the letters from *p* to *z.* It we want, as it is usually, to have infinitely many propositional variables, the employed notation using the letter *p* indexed by natural numbers is indispensable.• (the end of remark)

An arbitrary finite string of elements of $\mathbf{AL_{PC}}$ is an expression of **PC**. Since "good" expressions (well-formed formulas) of **PC**, should (roughly) correspond to correct grammatical sentences of ordinary language, we should accept expression like $\neg p_1$ or $p_2 \wedge p$ and reject strings like $\neg p_1) \vee (p_2$. It leads to the following definition of the set $\mathbf{FOR_{PC}}$ (the set of well-formed formulas, that is wffs, of **PC**).

(**D1**) $\mathbf{FOR_{PC}}$ (= $\mathbf{L_{PC}}$) is the smallest set satisfying the conditions:

(a) $p_i \in \mathbf{FOR_{PC}}$, for any $i = 1, 2, 3,...$;

(b) if $A \in \mathbf{FOR_{PC}}$, then $\neg A \in \mathbf{FOR_{PC}}$;

(c) if $A, B \in \mathbf{FOR_{PC}}$, then $A \wedge B, A \vee B, A \rightarrow B, A \leftrightarrow B \in \mathbf{FOR_{PC}}$.

**Remark 2**. $\mathbf{FOR_{PC}}$ is the smallest set satisfying the conditions (a) – (c) listed in (**D1**) in this sense that it is contained in every set conforming that conditions (or $\mathbf{FOR_{PC}}$ is equal to the intersection of all sets obeying (a) – (c)). We can also say that $\mathbf{FOR_{PC}}$ is the smallest set containing $\mathbf{VAR_{PC}}$ (that is, atomic formulas) and closed under (b) and (c). Another way of expressing this fact is to omit the expression "the smallest set" and add the condition (d) nothing else belongs to $\mathbf{FOR_{PC}}$ except strings satisfying (a) – (c). The letters $A$ and $B$ do not belong to $\mathbf{AL_{PC}}$. They are metalinguistic variables used in our description of **PC** (that is, belong to metalanguage) and refer to arbitrary wffs of this logic. (**D1**) is a typical example of inductive definition.●

Having the definition of of wffs, we can formulate **PC** as a deductive theory. We will work with **PC** in its (one of many) axiomatic codifications; other codifications are given by natural deduction rules, sequents, dyadic trees and other devices. Basically, we have to ways to proceed. The first is the Hilbert-style formalization which consists in using of axiom-schemata, and the second employs concrete fomulas from $\mathbf{AL._{PC}}$. I will illustrate both strategies. Let letters $A$, $B$, $C$ be metalinguistic variables referring to arbitrary formulas of **PC**. The following axioms (**AX1**) (due to Hilbert) can be adopted:

(**A1**) $A \rightarrow (B \rightarrow A)$

(**A2**) $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$

(**A3**) $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$

(**A4**) $A \wedge B \rightarrow A$

(**A5**) $A \wedge B \rightarrow B$

(**A6**) $(A \rightarrow B) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow B \wedge C))$

(**A7**) $A \rightarrow A \vee B$

(**A8**) $B \rightarrow A \vee B$

(**A9**) $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C))$

(**A10**) $(A \leftrightarrow B) \rightarrow (A \rightarrow B)$

(**A11**) $(A \leftrightarrow B) \rightarrow (B \rightarrow A)$

(**A12**) $(A \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow (A \leftrightarrow B)$

(**A13**) $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$

(**A14**) $A \rightarrow \neg\neg A$

(**A15**) $\neg\neg A \rightarrow A$

In order to obtain **PC**, we must supplement (**A1**) – (**A15**) by inference rules. It is suffocient to add modus ponens ((**MP**)) as the only inference device. It allows us to derive *B* from *A* together with $A \rightarrow B$.

A nice feature of this set of axioms is that we can easily distinguish subsets related to particular connectives. (**A1**) – (**A3**) characterize implication, (**A4**) – (**A6**) conjunction, (**A7**) – (**A9**) disjunction, (**A10**) – (**A12**) equivalence, and (**A13**) – (**A15**) negation. Now if we drop (**A15**), we obtain the axiom set for intutionistic logic. We can say that (**A1**) – (**A15**) characterize classical propositional logic, but (**A1**) – (**A14**) do the same job for intuitionistic propositional logic. Furthermore, (**A1**) – (**A2**) plus (**MP**) produce the positive implicational propositional calculus, that is the implicational part of intuitionistic propositional logic. If we add (**A3**), we obtain classical propositional calculus in based on implication as the sole logical constant), (**A1**) – (**A11**) plus (**MP**) give the positive logic (logic without negation), (**A1**) – (**A12**) together with the formula $(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A)$ and (**MP**) provide the minimal logic (minimal condition for implication consistent with intutionistic logic), and (**A1**) plus (**A3**) plus (**MP**) define the pure implicational logic, that is, the implicational part of the minimal logic. A surprising fact about positive logic is that it is weaker, if it is constructed independently of **PC** as compared with this same set of axioms extracted from classical propositional calculus.

As an axiomatization by concrete formulas we can take (after Łukasiewicz) the set (**AX2**) which is based on negation and implication as primitive propositional constants and consists of

(**A'1**) $(p_1 \rightarrow p_2) \rightarrow ((p_2 \rightarrow p_3) \rightarrow (p_1 \rightarrow p_3))$

(**A'2**) $(p_1 \rightarrow (\neg p_1) \rightarrow p_2)$

(**A'3**) $\neg p_1 \rightarrow (p_1 \rightarrow p_1)$.

plus (**MP**) and the substitution (**SR**) rule which licenses transformations of formulas by substituting variables by arbitrary wffs.

**Remark 3**. The fact that there are considerably less axioms on (**AX2**) than in **(AX1)** does not decrease the expressive power of the former, because it is known that implications and negation are sufficient to define all other connectives of **PC**. Thus, we have the definitions:

(a) $p_1 \wedge p_2 =^{df} \neg(p_1 \rightarrow \neg p_2)$;

(b) $p_1 \vee p_2 =^{df} \neg p_1 \rightarrow p_2$;

(c) $p_1 \leftrightarrow p_2 =^{df} (p_1 \rightarrow p_2) \wedge (p_2 \rightarrow p_1)$.

Of course, (**AX2**) does not admit any simple extraction of subsystems of classical logic by deleting some axioms.•

**Remark 4**. The clear difference between (**AX1**) and (**AX2**) consists in the stock of inference rules; **PC** based on **(AX2)** has additionally (**SR**). Of course, we can transform (**A1**) – (**A15**) into schemata as well as (**A'1**) – (**A'3**) into concrete formulas. However, such changes are not only calligraphic. Axiomatizations by schemata are always infinitistic in this sense that every formula falling under a scheme can be taken as a proper axiom. For instance, the formulas $p_1 \rightarrow (p_2 \rightarrow p_1)$ and $(p_1 \rightarrow p_1) \rightarrow ((p_2 \rightarrow p_2) \rightarrow (p_1 \rightarrow p_1))$ are instances of (**A1**). In general, every scheme of a formula generates a countably infinite set of concrete formulas. Note that the syntax of schemata is analogous as the syntax depicted by (**D1**), except the condition (a) of that definition. Although **PC** related to (**AX1**) is not finitary axiomatizable, this fact is not so dramatic as in the case of first-order arithmetic of natural numbers. The latter has even no recursive axiomatization, although the codification of **PC** by schemata is recursive. The same remark also applies to **PL.**•

**Remark 5**. Both axiomatizations basically yield the same stock of theorems of **PC** provided that . On the other hand, they have different metalogical properties (see below). Yet (**AX2**) transformed into schemata simplifies several metalogical considerations. In the due course, I will sometimes use the system given by the schemata (plus (**MP**))

(**A''1**) $(A \rightarrow B) \rightarrow ((A \rightarrow C) \rightarrow (A \rightarrow C))$;

(**A''2**) $A \rightarrow (\neg A \rightarrow B)$

(**A''3**) $(\neg A \rightarrow A) \rightarrow A$.

This set of schemata is denoted by (**AX3**)•

## §3. PC-theoremhood, proofs and derivations.

(**D2**) The set of **PC**-theorems is the smallest $\mathbf{TH^{PC}}$ set of **PC**-formulas such that

(a) if $A \in$( **AX1**), then $A \in \mathbf{TH^{PC}}$;

(b) if $A \in \mathbf{TH^{PC}}$ and $A \to B \in \mathbf{TH^{PC}}$, then $B \in \mathbf{TH^{PC}}$.

(**D3**) $A$ is a theorem derived by **PC** ( $\vdash^{\mathbf{PC}} A$) iff there is a finite sequence $\mathbf{S} = A_1, ..., A_n$ ($n$ = 1, 2, ...) of **PC**-formulas such that

(a) $A = A_n$;

(b) for any $i \leq n$, $A_i \in \mathbf{TH^{PC}}$, or

(c) there are formulas $A_j$ ($j < i$) and $A_k$ ($k < i$) such that $A_j = A_k \to A_i$, that is, $A_i$ is obtainable by (**MP**) from the formulas which are prior to it in **S**.

(**D2**) which operates, so to speak, inside **PC** calculus, may be generalize to

(**D4**) Assume that $X$ is a set of **PC**-formulas, that is, $X \subseteq \mathbf{L_{PC}}$.
**X** $\vdash A$ iff there is a finite sequence $\mathbf{S} = A_1, ..., A_n$ ($n$ = 1, 2, 3,...) of **PC**-formulas such that (a) $A_n = A$; (b) for any $i \leq n$, $A_i \in$ **(AX1)** or $A_i \in \mathbf{X}$ or there are **PC**-formulas $A_j$ ($j < i$) and $A_k$ ($k < i$) such that $A_j = A_k \to A_i$, that is, $A_i$ is obtainable by (**MP**) from the formulas which are prior to it in **S**.

(**D2**) defines the concept of proof of theorems of **PC**, whereas (**D3**) generates the concept of derivation from arbitrary set of sentences by deductive means of classical propositional calculus. Proofs and derivations are always relative to definite rules of inference, for example, (**MP**) and (**SR**). The expression **X** $\vdash A$ means "$A$ is derivable from **X**". Note that every axiom has automatically a proof which consists exactly of one line in which the axiom question is written. It is easy to see that every **PC**-theorem, in particular, every **PC**-axiom is derivable from every set.

**Remark 6**. The term "sequence" has in (**D2**) – (**D3**) somehow non-standard application. Strictly speaking, sequences are functions from the set of natural numbers to arbitrary sets of objects. Such functions connects natural numbers with some objects. However, when we are speaking about sequences of formulas, for instance, about **S** = $A_1, ..., A_n$ ($n$ = 1, 2, 3,...), we rather think about codomains of suitable functions. The connection between natural numbers and formulas is indicated by lower indices.•

### §4. The deduction theorem.

This theorem, discovered by Tarski in 1921 and, independently, by Herbrand in 1930 belongs to the most important metalogical results.

(**DT**) if $\mathbf{X} \cup \{A\} \vdash B$, then $\mathbf{X} \vdash (A \rightarrow B)$.

Proof (induction of the length of proof). We assume that there is a derivation of $B$ from $\mathbf{X} \cup \{A\}$.

I. The basis of induction.

The shortest derivation of $B$ from $\mathbf{X} \cup \{A\}$ consists of one line. It means that $B$ is a PC-axiom or $B = A$ or $B \in \mathbf{X}$. If $B = A$, then $(A \rightarrow B) = (A \rightarrow A)$. If $B$ is an axiom, so is $B \rightarrow (A \rightarrow B)$. By (**MP**), $(A \rightarrow B)$ belongs to consequences of arbitrary set, in particular $(A \rightarrow B) \in \mathbf{X}$. If If $B = A$, then $(A \rightarrow B) = (A \rightarrow A)$. Since $(A \rightarrow A)$ is a theorem, it belongs to consequences of arbitrary set, in particular $(A \rightarrow A) \in \mathbf{X}$. Finally, if we have that $B \in \mathbf{X}$, then the formula $B$ is derivable from **X** (by (**D3**c)), but the same concerns $B \rightarrow (A \rightarrow B)$ as well as $(A \rightarrow B)$.

II. The induction step.

Let $k$ be an arbitrary natural number greater than 1. Assume for induction that (**DT**) holds for every $i$ such that $i < k$. It means that if $B$ is provable in steps $S_1, ..., S_i$ from **X**. We will show that the same holds, if $B$ has the proof of the length $k$. If $B$ is an axiom or $B = A$ or $B \in \mathbf{X}$, we proceed like in the part (I). It remains to check the case when $B$ is derivable by (**MP**). It means that formulas $B_1$ and $B_i \rightarrow B$ are elements of the set $\{B_1, ..., B_{k-1}\}$. On the assumption which serves as the basis of induction, there are two following derivations: $A \rightarrow B_i$ and $A \rightarrow (B_i \rightarrow B)$. The application of the theorem of

**PC**: $A \rightarrow (B_i \rightarrow B) \rightarrow ((A \rightarrow B_i) \rightarrow (A \rightarrow B))$ and double use (**MP**) lead to derivation formula $A \rightarrow B$.

(**DT**) can be strengthened to equivalence. In order to prove the implication

(**DT'**) if **X** $\vdash (A \rightarrow B)$, then **X** $\cup \{A\} \vdash B$,

it is enough to assume the antecedent **X** $\vdash (A \rightarrow B)$, and further (a) to observe that there is a derivation of $A \rightarrow B$ from **X**, (b) that observe there is a derivation of $A \rightarrow B$ from **X** $\cup \{A\} \vdash B$, and (c) to use (**MP**). Thus, bot versions of the deduction theorem, namely (**DT**) and (**DT'**) give

(**DT''**) **X** $\cup \{A\} \vdash B$ iff **X** $\vdash (A \rightarrow B)$.

(**DT**) provides a simple justification for the following fact

(**F**) if $A$ is a theorem of **PC**, then $\varnothing \vdash A$.

It means that every theorem of **PC** is derivable form the empty set of premises. (**D3**) justifies, for example, that $\{A\} \vdash A$. Applying (**DT**) we obtain $\vdash (A \rightarrow A)$ and it exactly means that $\varnothing \vdash (A \rightarrow A)$. It is easy to justifies that the elements of (**AX3**) satisfy (**F**) and (**MP**) preserves this property. Thus, every logical theorem is transformable into a rule of inference and vice versa. This establishes the exact parity between logic (in this case, the matter concerns **PC**) conceived as rules of inference and logic as a set of theorems. On the other hand, rules are distinguished in a sense, because, except a trivial situation in which we take all theorems as rules, every axiomatization requires ruled for generating further theorems.

**Remark 7**. It is important to note that the deduction theorem is not applicable to formalization of **PC** with (**SR**). Assume that it is and that **X** $\cup \{p_1\} \vdash p_2$. By (**SR**) we could derive any formula from the set **X** $\cup \{p_1\}$, in particular, the formula $\neg p_1$. Now assume that **X** $= \{p_1\}$. By (**DT**), applied two times, we obtain that $\varnothing \vdash p_1 \rightarrow (p_1 \rightarrow \neg p_1)$. However, this last formula is not a theorem. It shows that some properties of **PC** are dependent on the chosen formalization. If one like to save (**DT**) for formalization with (**SR**), one must handle formulas with all their possible substitutions.•

Another important feature of (**DT**) is that its validity is equivalent to the following simple condition: The deduction theorem holds for every logical system in which the formulas $A \to (B \to A)$ and $(A \to B) \to ((B \to C) \to (A \to C))$ are theorems. Now observe that both formulas belongs (**AX1**) and characterize positive implication. Thus (**DT**) does the same job.

## §5. Consequence operations.

Having the definition of derivability, we can define an important concept of logical consequence. The definition is this:

(**D5**) $A \in Cn(X)$ iff $X \vdash A$.

Although *Cn* (the consequence operation) and $\vdash$ (the consequence operator) are mutually interdefinable, there is a categorial difference between them. Reminding that **L** is a language understood as a set of formula, *Cn* is a mapping from $2^{\mathbf{L}}$ to $2^{\mathbf{L}}$ that transforms sets of formulas into sets of formulas, and the consequence operator maps $2^{\mathbf{L}}$ into **L**, that is, sets of formulas are transformed into single formulas. Since (**D4**) is applicable to any language, I omitted the index pointing out that **PC** is considered.

It is also possible to take the concept of logical consequence as primitive one and establish its properties by axioms. Since we have infinitely many (in fact, uncountably many, even if **L** is a countably infinite language) mappings from $2^{\mathbf{L}}$ to $2^{\mathbf{L}}$, we need to accept some constraints selecting a "reasonable" consequence operation (or operations). It was due to Tarski who characterized the consequence operation associated with the propositional calculus by the following axioms (he considered *Cn* as a closure operation modelled by some intuitions derived from topology):

(**C1**) $\varnothing \leq \mathbf{L} \leq \aleph_0$

(**C2**) $X \subseteq CnX$

(**C3**) if $X \subseteq Y$, then $CnX \subseteq CnY$

(**C4**) $CnCnX = CnX$

(**C5**) if $A \in CnX$, then $\exists Y \subseteq X \wedge Y \in$ **FIN**$(A \in CnY)$

(**C6**) if $B \in Cn(X \cup \{A\})$, then $(A \rightarrow B) \in CnX$

(**C7**) if $(A \rightarrow B) \in CnX$, then $B \in Cn(X \cup \{A\}$

(**C8**) $Cn\{A, \neg A\}$ = **L**

(**C9**) $Cn\{A\} \cap Cn\{\neg A\} = \varnothing$

We can divide the axioms (**C1**) – (**C9**) into three groups. The first group includes (**C1**) – (**C5**) as general axioms for *Cn*. (**C1**) says that the cardinality of **L** is at most denumerably (denumerably – finitely or so many as natural numbers) infinite; (**C2**) that any set is a subset of the set of its consequences; (**C3**) establishes the monotonicity of *Cn*; (**C4**) its idempotency; (**C5**) states the finiteness condition which means that if something belongs to *Cn*(*X*), it may be derived from a finite subset of *X*. In other words: every inference is finitary, that is, it is possible to perform it on the base of a finite set of premises and, according to the character of rules, finitely long. (**C1**) – (**C5**) do not provide any logic in its usual sense. The logical machinery is encapsulated by the rest of axioms (related to logic based on negation and); (**C6**) is simply (**DT**) and (**C7**) provides a version of (**MP**); and both axioms establish the meaning of classical implication. (**C8**) – (**C9**) characterize negation. Of course, we can also formulate *Cn's* suitable for positive, minimal or intutionistic logic. We can also prove that *A* is a theorem of **PC** if it is the consequence of the empty set.

### §6. Semantics of PC.

We introduce two objects **1** (truth) and **0** (falsity) as logical values and adopt that **~1** = **0** as well as **~0** = **1** (the sign ~ denotes an operation on logical values which must be distinguished from negation as a connective); in fact, we assume that the algebra of logical values is Boolean. An evaluation of variables is any function **v**: $\mathbf{VAR_{PC}} \Rightarrow \{\mathbf{1}, \mathbf{0}\}$, that is, such a function which transforms variables into logical values. Now this function is extendible to other mapping (evaluation), namely **V**: $\mathbf{FOR_{PC}} \Rightarrow \{\mathbf{1}, \mathbf{0}\}$ which transforms **PC**-formulas into the set {**1**, **0**} of logical values. The formal definition is as follows:

(**D6**) (a) $\mathbf{V}(p_i) = \mathbf{v}(p_i) = \mathbf{1}$ or $\mathbf{0}$;

(b) $\mathbf{V}(\neg A) = \ \sim\mathbf{V}(A)$;

**(c)** $\mathbf{V}(A \wedge B) = \mathbf{1}$ iff $\mathbf{V}(A) = \mathbf{V}(B) = \mathbf{1}$; otherwise $\mathbf{V}(A \wedge B) = \mathbf{0}$;

(d) $\mathbf{V}(A \vee B) = \mathbf{1}$ iff $\mathbf{V}(A) = \mathbf{1}$ or $\mathbf{V}(B) = \mathbf{1}$; otherwise $\mathbf{V}(A \vee B) = \mathbf{0}$;

(e) $\mathbf{V}(A \rightarrow B) = \mathbf{1}$ iff $\mathbf{V}(A) = \mathbf{0}$ or $\mathbf{V}(B) = \mathbf{1}$;
otherwise $\mathbf{V}(A \rightarrow B) = \mathbf{0}$;

(f) $\mathbf{V}(A \leftrightarrow B) = \mathbf{1}$ iff $\mathbf{V}(A) = \mathbf{V}(B) = \mathbf{1}$ or $\mathbf{0}$;
otherwise $\mathbf{V}(A \leftrightarrow B) = \mathbf{0}$.

**Remark** 8. The definition (**D5**) reproduces familiar truth-tables used for checking theorems of **PC**. Since every **PC**-formula is a finite string of symbols taken from $\mathbf{AL_{PC}}$, (**D5**) provides a finitistic procedure (a decision procedure) for solving the problem of the theoremhood of **PC**. However, this victory is Pyrronian. Since for (very) very long formulas the procedure is not realizable in polynomial time and requires exponential time, this decision method is practically not accessible in general and its universal applicability is only a theoretical advantage. Hence, deductive proofs of theorems are more feasible, even in cases of formulas which have more variables (the author claims that more than 4, but it is a subjective matter).●

Finally we define

(**D7**) (a) $A$ is **PC**-satisfiable (has a model) iff for some **v**, $\mathbf{V}(A) = \mathbf{1}$**;**

(b) **X** is **PC**-satisfiable iff for every $A \in \mathbf{X}$, $A$ is **PC**-satisfiable;

(c) $A$ is a **PC**-tautology ($A \in \mathbf{TL_{PC}}$, $\models A$) iff for every **v**, $\mathbf{V}(A) = \mathbf{1}$;

(d) $A$ is a semantic consequence of **X** ($\mathbf{X} \models A$) iff for every **V**, if **X** is satisfied under **V**, so is $A$ as well;

(e) $A$ is a **PC**-contradiction iff $A$ is not satisfiable, that is, for any **v**, $\mathbf{V}(A) = \mathbf{0}$.

This definition proposes to regard tautologies as formulas true under every evaluation of their variables. The concept of "being a theorem" is syntactic, whereas the concept of "being tautology" is semantic one. Similarly, the symbol $\models$ refers to a semantic notion, whereas the consequence operator ($\vdash$) is syntactic in its essence. (**D6**) suggests that truth-tables provide also a method showing how to solve the problem of

**PC**-tautologicity. However, it is important to prove independently of definitions that the syntactic concept of **PC-**theoremhood and the semantic concept of **PC**-tautologicity are coextensional. It is the principal content of the completeness theorem (see below). one of the most important statements concerning logic, because its provability or not throws the light how both kinds of concepts are mutually related in the given case.

**§7. Further metalogical concepts**.

The definitions given below are applicable to any sets of sentences formulated in a given language **L**, not only to language of propositional calculus.

(**D8**) **X** is a deductive system (**X** $\in$ **SYS**) iff $Cn\mathbf{X} \subseteq \mathbf{X}$;

(**D9**) **X** is (finitely, recursively) axiomatizable (**X** $\in$ **AX**) iff for some (finite, recursive) set $\mathbf{Y} \subseteq \mathbf{X}$, $Cn\ \mathbf{Y} = \mathbf{X}$;

(**D10**) **Y** is an independent set of axioms for **X** iff (a) $\mathbf{Y} \subseteq \mathbf{X}$ ; (b) $Cn\mathbf{Y} = \mathbf{X}$; (c) for any $A \in \mathbf{Y}$, $A \notin Cn(\mathbf{Y} - \{A\}$;

(**D11**) **X** is absolutely consistent (**X** $\in$ **ACON**) iff $Cn\mathbf{X} \neq \mathbf{L}$;

(**D12**) **X** is negation consistent (**X** $\in$ **NCON**) iff for any $A$, $A \wedge \neg A \notin Cn\mathbf{X}$.

(**D13**) **X** is syntactically complete (**X** $\in$ **SYNCOM**) iff for any $A$, $A \in Cn\mathbf{X}$ or $\neg A \in Cn\mathbf{X}$;

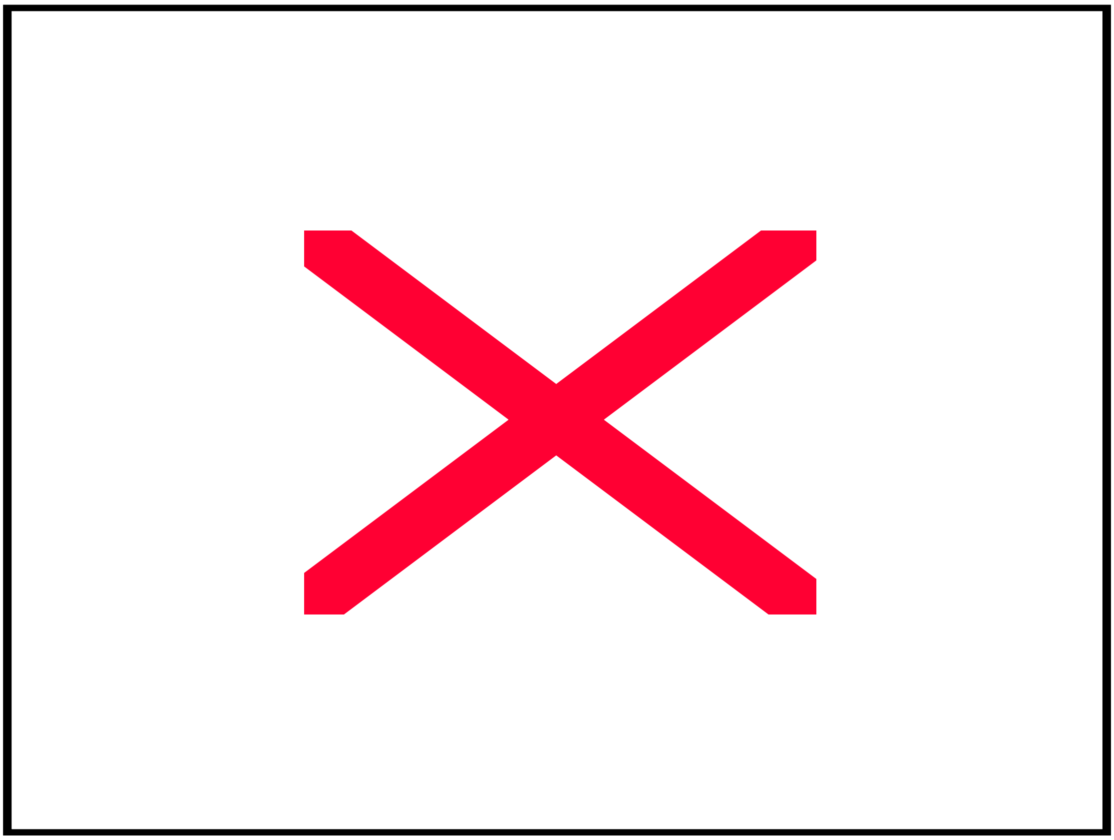
(**D14**) (strong semantic completeness) **X** is strongly semantically complete (**X** $\in$ **SSEMCOM**) iff for any $A$, $\mathbf{X} \vdash A$ iff $\mathbf{X} \models A$;

(**D15**) (weak semantic completeness) $\vdash A$ iff $\models A$;

(**D16**) **X** is Post-complete or maximally consistent (**X** $\in$ **PCOM**) iff $\mathbf{X} \in \mathbf{ACON}$ and for any $A \notin \mathbf{X}$, $Cn\{\mathbf{X} \cup \{A\} = \mathbf{L}$.

(**D17**) **X** is decidable (**X** $\in$ **DEC**) iff there is an algorithmic (mechanical) procedure which solves the problem whether for any $A$, $A \in \mathbf{X}$ or $A \notin \mathbf{X}$.

The last definition is informal one. However, if the Church thesis which identifies the concept of mechanical procedure with the concept of recursivity, is



accepted, decidability becomes well defined in the mathematical sense. (**D8**) may be strengthen to equivalence, because (from the definition of *Cn*) we have that $\mathbf{X} \subseteq Cn\mathbf{X}$. Instead "system" the term "theory" can be used in this context. (**D7**) expresses the fact that **X** is theory if and only if **X** is closed under the consequence operation. It means that consequences of theories belong to them. Semantic completeness is in fact the implication: if $\mathbf{X} \models A$, then $\mathbf{X} \vdash A$ (roughly speaking: every truth is provable). The implication "if $\mathbf{X} \vdash A$, then $\mathbf{X} \models A$" expresses the property of soundness (every theorem is true). (**D8**) – (**D13**), (**D16**) and (**D17**) define syntactic concepts; (**D13**) and (**D14**) deal with semantic notions. Another definition of semantic completeness (the Gödel-Malcev completeness) is:

(**D18**) **X** is semantically complete iff **X** has a model.

**PC** is recursively axiomatized by (**AX1**) or (**AX3**), and (**AX2**) is its finite axiomatization. (**AX1**) – (**AX3**) are independent axiomatizations of **PC**. Further, **PC** is absolutely and negation consistent. In fact, both concepts of consistency are equivalent in **PC** (and in every system having negation is its language). Thus, since typical formalizations are performed in language in which negation occurs, we will simply speak about consistency (**CONS**). In order to prove consistency of **PC**, we first adopt an evaluation **v**, such that $\mathbf{v}(p_i) = \mathbf{1}$, for every $i$; of course, such an evaluation exists. Then, we observe that this evaluation is also an model for all axioms and is also preserved by (**MP**). So every theorem of propositional calculus is satisfied by our valuation **v**. **PC** is sound because its axioms are universally satisfied and this property is preserved by modus ponens. Observe also that if a set is satisfiable, its every finite subset is also satisfiable; the reverse connection (compactness) is more complicated and follows from the completeness theorem (more exactly, from the form related to (**D18**)). **PC** syntactically incomplete because no variable or its negation is a **PC**-theorem. **PC** is decidable (see above). It is also Post-complete, but only in its formalization by concrete formulas, not when we use schemes. It is the next example (see remark 7 above) of how metalogical properties are sometimes dependent on the manner in which logical systems are formalized. Assume that we add $p_i$ as an axiom of **PC**. Having **(SR)** we can prove any formula of **PC** as its theorem. Note that provability (derivability) defined for $\vdash$ generates a consequence operation related only to (**MP**). So the consequence operation given by (**MP**) and (**SR**) is essentially different than *Cn* axiomatized by (**C1**) – (**C9**). Assume that **PC** formalized by scheme would be Post-complete. Take $p_1$ as $A$ and add it to (**AX1**). By the axiom $A \rightarrow (B \rightarrow A)$ and (**MP**), the formula $B \rightarrow p_1$ would be a theorem of **PC** for any formula represented by $B$, but it is not true. Thus, **PC** in its schematic (Hilbertian) formulation is not Post-complete. On the other hand, it has a related property, namely

(**PCOM'**) $Cn\{A^*\} = \mathbf{L}_{\mathbf{PC}}$, where $A^*$ represents non-theorems of **PC**.

Although ways of formalization influence properties of logics, there are, on the other hand, counterparts of a particular feature of a given logical system in systems governed by other formal devices. For instance, when we suitably restrict substitutions, we can also save (**DT**) for **PC** in its version related to (**AX2**). The completeness theorem in its various versions will be discussed in §8.

### §7. The Lindenbaum theorem on maximalization.

(**L**) Every consistent set of **PC-**formulas has its maximal extension:

if $\mathbf{X} \in \mathbf{CON}$, then there is a set **Y** such that $\mathbf{X} \subseteq \mathbf{Y}$ and $\mathbf{Y} \in \mathbf{PCON}$.

Proof. Let **X** be a consistent set of sentences. Since the set $\mathbf{FOR_{PC}}$ is countable, we can form a sequence $<A_1, A_2, A_3, ...>$ of formulas. Now we define an infinite sequence of sets (of **PC**-formulas) $\mathbf{X_0}$, $\mathbf{X_1}$, $\mathbf{X_2}$, ... such that (a) $\mathbf{X} = \mathbf{X_0}$, (b) $\mathbf{X_{n+1}} = \mathbf{X_n} \cup A_{n+1}$ iff $\mathbf{X_n} \cup A_{n+1} \in \mathbf{CON}$; otherwise $\mathbf{X_{n+1}} = \mathbf{X_n}$. We will prove that the union of sets

$$\mathbf{X^*} = \mathbf{X_0} \cup \mathbf{X_1} \cup \mathbf{X_2} \cup ...$$

is maximally consistent, that is, $\mathbf{X^*} \in \mathbf{PCON}$. At first, it is to be demonstrated that $\mathbf{X}^*$ is consistent. It follows from the construction of the sequence $\mathbf{X_0}$, $\mathbf{X_1}$, $\mathbf{X_2}$, ... that for any $i$ ($i$ = 1, 2, ...), $\mathbf{X_i} \in \mathbf{CON}$. Assume that $\mathbf{X}^*$ is inconsistent. Thus we have $\mathbf{X}^* \vdash A$ as well as $\mathbf{X^*} \vdash \neg A$. These derivations consists of finitely many formulas. We take $A_k$ as a formula having the greatest number in our enumeration of all formulas which occurs in both derivations, that is, $\mathbf{X^*} \vdash A$ and $\mathbf{X^*} \vdash \neg A$. So $\mathbf{X_k} \vdash A$ and $\mathbf{X_k} \vdash \neg A$. Thus, $\mathbf{X_k}$ is inconsistent, but it contradicts assumption that $\mathbf{X_i} \in \mathbf{CON}$, for any $i$. It proves that $\mathbf{X_k}$ must be consistent. In order to justify that $\mathbf{X}^* \in \mathbf{PCON}$, we first take an arbitrary formula $A_k$ where $k$ pointing out the place of $A_k$ in our enumeration of formulas. Assume that $A_k \notin \mathbf{X^*}$. If so, $\mathbf{X_{k-1}} \cup \{A_k\}$ has to be inconsistent because in the opposite case we would have: $\mathbf{X_{k-1}} \cup \{A_k\} = \mathbf{X_k}$. It entails that for some formula $B$, $\mathbf{X_{k-1}} \cup \{A_k\} \vdash B$ and $\mathbf{X_{k-1}} \cup \{A_k\} \vdash \neg B$. Furthermore, if $\mathbf{X_{k-1}} \subseteq \mathbf{X}^*$, then $\mathbf{X^*} \cup \{A_k\} \vdash B$ and $\mathbf{X}^* \cup \{A_k\} \vdash \neg B$. However, either $A_k \in \mathbf{X}^*$ or $\mathbf{X}^* \cup \{A_k\}$ is inconsistent and it means that $\mathbf{X^*} \in \mathbf{PCON}$ .

**Remark 9**. (**L**) belongs to the most impressive results in metalogic. It is a syntactic but non-constructive theorem, because it establishes the existence of a set which is not effectively constructible. It is so, because the construction of the sequence of sets $\mathbf{X_0}$, $\mathbf{X_1}$, $\mathbf{X_2}$, ... is not base on an effective prescription. (**L**) belongs to the family of so called maximal principles, like the Kuratowski-Zorn lemmas in set theory) and, as we will see below, has the crucial application in proving the completeness theorem by the Henkin method.●

**Remark 10**. It is not so that there is the unique maximally consistent set relative to a given set **X**.•

**§8. The completeness theorem for PC.**

Semantic completeness of **PC** in the sense of (**D14**) was first proved by Emil Post. The sketch of his proof is as follows. Let $A$ be an arbitrary formula. It has a (effectively calculable) disjunctive normal form $dsA$ such that $\vdash (dsA \leftrightarrow A)$. There exists an effective method of proving every $dsA$ such that $A$ is a tautology. Assume that $A \in$ **TL**. If $A$ is in $dsA$, it is known how to prove it. On the other hand, if $A$ is not in $dsA$, we have a method how to find its $dsA$ such that $\vdash (dsA \leftrightarrow A)$. Since we have (a) $\vdash dsA$ and (b) $\vdash (dsA \leftrightarrow A)$, we have also $\vdash A$, for arbitrary $A$. Hence every **PC**-tautology is provable.

Now we will prove the Gödel-Malcev completeness theorem for **PC**

(**GMCT**) Every consistent set of **PC**-formulas has a model.

Proof.

First we state (without proofs) three lemmas:

(**Le1**). The following are **PC**-theorems:

(a) $A \to (B \to A)$;

(b) $\neg A \to (A \to B)$;

(c) $A \to (\neg B \to \neg(A \to B))$;

(**Le2**). If **X** is a maximally complete, then for every $A$, either $A \in$ **X** or $A \notin$ **B**.

(**Le3**). For every **X** $\in$ **PCOM**, if **X** $\vdash A$, then $A \in$ **X**.

The essential part of the proof. Assume that **X** $\in$ **CON**. By (**L**), **X** has a maximal consistent extension $\mathbf{X}^*$. The proof consists in showing that if $\mathbf{X}^*$ has model, then **X** also has a model. We adopt the interpretation such that $\mathbf{v}(p_i) = \mathbf{1}$, for any $i$ which indexes atomic formulas belonging to $\mathbf{X}^*$; otherwise $\mathbf{v}(p_i) = \mathbf{0}$. Now we will show that

(**L'**) assuming **v**, $V(A) = \mathbf{1}$ iff $A \in \mathbf{X^*}$.

Proof of (**L'**) uses induction with respect to the number of connectives in $A$. For simplicity we will use (**AX3**).

I. Basis.

$A = p_i$.

(**L'**) is true in this case in virtue of definition of **v**.

II. II. Induction step.

III. Assume that (**L'**) holds for any formula which has less than $m$ connectives. We must consider two cases: (a) $A = \neg B$; and (b) $A = B \rightarrow C$, where $B$, $C$ have less connectives than $m$.

IV. Case (a) (If-part) Assume that $\mathbf{V}(A) = \mathbf{1}$. Since, $A = \neg B$ , $\mathbf{V}(B) = \mathbf{0}$. In virtue of inductive assumption $B \notin \mathbf{X^*}$ (it has less connectives than $\neg B$). By (**Le2**), $A \in \mathbf{X^*}$.

(Only-part) Assume that $A \in \mathbf{X^*}$. So $B \notin \mathbf{X^*}$ and $\mathbf{V}(B) = \mathbf{0}$. Hence, $\mathbf{V}(A) = \mathbf{1}$.

Case (b) (If-part) Let $\mathbf{V}(A) = \mathbf{1}$. Since $A = B \rightarrow C$, either $\mathbf{V}(A) = \mathbf{0}$ or $\mathbf{V}(C) = \mathbf{1}$. Hence, either $B \notin \mathbf{X^*}$ or $C \in \mathbf{X^*}$. Assume that $B \notin \mathbf{X^*}$. Hence, $\mathbf{X^*} \vdash \neg B$ ((**le3**). In virtue of lemma 1(c) and (**MP**), $\mathbf{X^*} \vdash B \rightarrow C$. So $A \in \mathbf{X^*}$. Assume that $C \in \mathbf{X^*}$. So $\mathbf{X^*} \vdash C$. In virtue of (**Le1**(a)) and (**MP**), $\mathbf{X^*} \vdash B \rightarrow C$. So $(B \rightarrow C) \in \mathbf{X^*}$, and we finally conclude that $A \in \mathbf{X^*}$.

(Only-part) Assume that $\mathbf{V}(A) = \mathbf{0}$. So $\mathbf{V}(B) = \mathbf{1}$ and $\mathbf{V}(C) = \mathbf{0}$. It means that $B \in \mathbf{X^*}$, but $C \notin \mathbf{X^*}$. Hence, $\mathbf{X^*} \vdash B$ and $\mathbf{X^*} \vdash \neg C$. Applying (**Le1**(c)) and (**MP**) (two times), we obtain $\mathbf{X^*} \vdash \neg(B \rightarrow C)$. Thus, $\neg(B \rightarrow C) \in \mathbf{X^*}$, $\neg A \in \mathbf{X^*}$ and $A \notin \mathbf{X^*}$. Hence, if $\mathbf{V}(A) = \mathbf{0}$, $A \notin \mathbf{X^*}$. By contraposition, if $A \in \mathbf{X^*}$, then $\mathbf{V}(A) = \mathbf{1}$. It completes the proof that **X*** has a model and, *a fortiori*, that every consistent set of sentences has a model. This completes the proof of (**GMCT**).

Assume that **X** has a model and is inconsistent. So we have $\mathbf{X} \vdash A$ as well as $\mathbf{X} \vdash \neg A$. Since **X** has a model, both $A$ and $\neg A$ are satisfied in this evaluation. However, it is impossible, because no **v** satisfies a contradiction. Hence, an inconsistent set of sentences has no model. This fact allows us to strengthen (**GMCT**) to equivalence:

(**GMCT'**) **X** is a consistent set of **PC**-formulas iff it has a model.

Having (**GMCT'**) and (proof omitted)

**(Le4)** $\mathbf{X} \cup \{\neg A\}$ is inconsistent iff $\mathbf{X} \vdash A$,

we can prove the strong completeness theorem: $\mathbf{X} \models A$ iff $\mathbf{X} \vdash A$. Assume that $\mathbf{X} \models A$. Observe that the set $\mathbf{X} \cup \{\neg A\}$ is not consistent and has no model. Hence, it is inconsistent. By **(Le4)**, $\mathbf{X} \vdash A$. If we take $\varnothing$ as $\mathbf{X}$, we obtain the weak completeness theorem: $\models A$ iff $\vdash A$.

The compactness theorem

(**CO**) If every finite subset of the set **X** of **PC**-formulas has a model, then **X** has a model,

is another very important consequence of (**GMCT'**). In order to prove (**CO**), we first observe that (the compactness of consistency)

(**CC**) If every finite subset of the set **X** of **PC-**formulas is consistent, then **X** is consistent.

Assume that every finite subset of **X** is consistent, but **X** is inconsistent. Thus, for some $A$, $\mathbf{X} \vdash A$ and $\mathbf{X} \vdash \neg A$. It means that there are derivations of $A$ and $\neg A$ from **X**. Both derivations must be finite. Hence, there is a finite subset **Y** of **X** such that $\mathbf{Y} \vdash A$ and $\mathbf{Y} \vdash \neg A$. Of course, **Y** is inconsistent, but it contradicts assumption that every finite subset of **Y** is consistent. (**CT**) follows from (**GMCT'**) and (**CC**).

**Remark 11**. Post's proof of the completeness theorem is purely syntactical. In fact, there more syntactic proofs of the completeness property of **PC**. Moreover, completeness in the sense that every tautology is provable is equivalent in **PC** to Post-completeness. It shows that semantic and syntactic concepts are perfectly equivalent in **PC**. This feature of propositional logic is somehow obscured by the proof of (**GMCT**) which is semantic and non-constructive. Post's proof (and other syntactic proofs) is not sufficient for proving strong completeness and compactness. It is perhaps interesting to note (for philosophy of logic) that although syntactic and semantic aspects of **PC** are equivalent in the object language, metalogical semantic methods of proof are stronger than syntactic ones.•

**Remark 12**. Now it is clear that *Cn* is so chosen that it saves soudness. In an informal sense, a given inference is sound, if it always leads from true premises to true conclusions. In other words, sound inferences are truth-preserving. It shows that axioms for *Cn* are based on definite pragmatic intuitions. However, it is also possible to define consequence operations which preserve falsity, that is, never produce true conclusions from false premises (dual consequence operation, rejection consequence operation; both introduced by Polish logicians).•

### §9. Predicate logic: syntax.

**Remark 13**. In order to simplify considerations, we will consider **PL** without functions symbols and identity. This restriction does not matter very much metalogical features of **PL**. In particular, all metatheorems about **PL** discussed below also hold for the system with function symbols and identity. Brackets will be used informally, like in the case of **PC**. The definitions of free and bound variables as well as correct substitutions for variables are assumed.•

The alphabet of **PL** ($\mathbf{AL_{PL}}$) consists of the following kinds of symbols:

Individual variables (**IVAR**): $x_1, x_2, x_3,...$;

Individual constants (**ICN**): $a_1, a_2, a_3,...$;

Predicate letters (**PREL**): ${}_1P_1, {}_1P_2, {}_1P_3,...$;

${}_2P_1, {}_2P_2, {}_2P_3,...$;

${}_3P_1, {}_3P_2, {}_3P_3,...$;

Connectives: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$;

Quantifiers: $\forall$(name: universal quantifier, reading: for every),

$\exists$ (name: existential quantifier, reading: there is).

If an expression is an individual variable or an individual constant, it belongs to terms. So the set **TER** = **IVAR** $\cup$ **ICN**. Predicate letters have various arities (depending on the number of their arguments, that is, terms). In general, the symbol ${}_iP_j$ refers to a predicate which has *i* arguments and stands in the list of all *i*-ary predicates at the place *j*. Quantifiers are new logical constants of **PL**. In fact, it is enough to introduce only one quantifier, because other is definable: $\forall$ as $\neg\exists\neg$, and $\exists$ as $\neg\forall\neg$. Also we can reduce the number of connectives but these details are not important here. However, we will use all connectives and both quantifiers.

(**D19**) **FOR$_{PL}$ (= L$_{PL}$)** is the smallest set satisfying the conditions:

(a) ${}_iP_j(t_1,..., t_i) \in$ **FOR$_{PL}$** , where $t_1,..., t_i \in$ **TER**;

(b) if $A \in$ **FOR$_{Pl}$**, then $\neg A \in$ **FOR$_{PL}$**;

(c) if $A, B \in$ **FOR$_{PL}$**, then $A \wedge B, A \vee B, A \rightarrow B, A \leftrightarrow B \in$ **FOR$_{PL}$**;

(d) if $A \in$ **FOR$_{PL}$**, then $\forall x_iA, \exists x_iA \in$ **FOR$_{PL}$**.

(**D20**) A sentence is a formula without free variables.

(**D21**) An atomic formula **(At)** is a formula defined in (**D19**a)

**§10. Predicate logic: deductive apparatus.**

The variety of deductive codifications of **PL** is considerably lesser than those of **PC**. It is related to properties of quantifiers. Also metalogical properties of **PL** are more absolute than in the case of propositional calculus, in the sense that they are less dependent on the chosen formalism. One of codifications is as follows (**AX4**):

(**AX1**) or (**AX3**) expressed in **L$_{PL}$**;

(**APL1**) $\forall x_iA \rightarrow A(t_i/x_i)$, if the term $t_i$ is substitutable in $A$ for $x_i$;

(**APL2**) $\forall x_i (A \rightarrow B) \rightarrow (A \rightarrow \forall x_i B)$, if $x_i$ is not free in $A$;

Rules of inference: (**MP**) suitably adapted for **L$_{PL}$** and the rule of generalization:

(**RG**) if $A$, then $\forall x_i A$.

The definitions (**D2**) – (**D4**) (being a proof, being a theorem, being a derivation) are directly adaptable to **PL** by adding the corresponding clauses referring to (**APL1**) – (**APL2**) and (**RG**). **(DT**) for (**PL**) requires its restriction to sentences, that is, formulas without free variables. The full proof continues cases beyond **PC**, that is, with respect to the axioms (**APL1**) – (**APL2**) and the rule **(RG**). However, the assumption that we deal only with sentences makes the proof almost trivial. Take

(**RG**), for example. Since $A$ is a sentence, its universal generalization is equivalent with it. So we need to prove that if $A$, then $A$. Also it is easily to extend axioms for the consequence operation in order to obtain $Cn$ for **PL** ((**C6**)) is to be restricted to sentences) by adding:

(**C10)** $A(t_i/x_i) \in Cn\{\forall x_i A\}$, if the term $t_i$ is substitutable in $A$ for $x_i$;

(**C11**) $(A \rightarrow \forall x_i B) \in Cn\ \{\forall x_i (A \rightarrow B)\}$, if $x_i$ is not free in $A$;

(**C12**) $\forall x_i A \in Cn\{A\}$.

### §11. Predicate logic: semantics.

The last section suggests that the matters of **PL**-syntax are a simple continuation of syntactic principles established for **PC**. The situation differs when we come to semantics. We assume that the semantic interpretation of connectives is given by rules introduced for **PC** (see (**D6**) above). Thus, it remains to define the interpretation for other expressions occurring in **AL<sub>PL</sub>**. In order to do it, we need to define the range of variability for individual variables and denotations for individual constants and predicate letters. Roughly speaking, an establishes non-empty set **U** of objects is the range of variability (the universe of discourse, the carrier of interpretation) for individual variables, distinguished objects from **U** are denotations of constants, subsets of **U** are denotations of unary predicates and ($n > 1$) subsets of $n$-ary ($n > 1$) Cartesians products defined on **U** are denotations of $n$-ary predicates. Intuitively, subsets of **U** are identified with properties of objects belonging to **U** and subsets of $n$-ary Cartesian products correspond with relations holding between those objects.

**Remark 14.** We can formally simplify the treatment by taking a subset of **U** as one termed Cartesian product. In this case, denotations of predicates become subsets of $n$-ary Cartesian products, where $n \geq 1$. However, the distinction between properties and relations seems to be important and should be preserved, even from a formal point of view.•

Formally speaking, the interpretation **I** of $\mathbf{L_{LP}}$ is a sequence

$$< \mathbf{U}, \mathbf{I}(a_1), \mathbf{I}(a_2), \mathbf{I}(a_3),\ldots, \mathbf{I}({}_1P_1),\ldots, \mathbf{I}({}_iP_j),\ldots >, \text{ where } \mathbf{U} \neq \varnothing,$$

$$\mathbf{I}(a_i) = \mathbf{u_i} \in \mathbf{U}, \mathbf{I}({}_1P_j) = \mathbf{U_j} \subseteq \mathbf{U}, \mathbf{I}({}_iP_j) = {}_\mathbf{i}\mathbf{R_j} \subseteq \mathbf{U} \times ... \times \mathbf{U} \ (i \text{ times}).$$

**Remark 15.** Nothing was said about the interpretation of quantifiers. In fact, it is a delicate matter. Sometimes $\mathbf{I}(\forall)$ is identified with **U**, and $\mathbf{I}(\exists)$ with arbitrary non-empty subsets of **U**. This treatment considers quantifiers as predicates. Another way uses cylindric algebras. In the most common view, quantifiers are interpreted according to their use and not by ascribing them set-theoretic constructs.•

The function **I** plays a similar role as **v** in semantics of **PC** in this sense that it provides semantic evaluation of the simplest elements of $\mathbf{AL_{PL}}$, namely terms, that is, variables and constants. Semantics for atomic and complex formulas is formulated with help of infinite sequences of objects. Let $\mathbf{s} = <\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, ...>$. All next definitions will establish semantic values (I will use the letter **V** also in the present context) relatively to sequences of objects.

**(D21)** (a) $\mathbf{V}(x_i, \mathbf{s}) = \mathbf{I}(x_i)$ and $\mathbf{V}(x_i) \in \mathbf{U}$

(b) $\mathbf{V}(a_i, \mathbf{s}) = \mathbf{I}(a_i) = \mathbf{u_i} \in \mathbf{U}$

**Remark 16.** We can read (**D21**a) in this way: the value of the variable $x_i$ at the sequence **s** is equal to an element of **U**, and (**D21**b): the value of the constant $c_i$ is equal to the element $\mathbf{u_i}$. This reading shows how constants differ from variables.•

The next definition uses the concept of satisfaction of a formula, according to (**D19**), in a model **M**. A model is a structure

$$\mathbf{M} = < \mathbf{U}, \mathbf{V}(x_i, \mathbf{s}), \mathbf{V}(x_2, \mathbf{s}), ..., \mathbf{V}(a_1, \mathbf{s}), \mathbf{V}(a_2, \mathbf{s}), ..., \mathbf{I}({}_1P_1), ..., \mathbf{I}({}_iP_j), ... >.$$

**Remark 17.** The interpretation **I** concerns the alphabet of our language. Sometimes it is called a model of a language. The terminology is of course optional, but it should help to avoid confusions. The term “model” has different and more important application, when we pass to definitions of satisfaction and truth. This difference is indicated by replacing **I** by $\mathbf{V}(E, \mathbf{s})$ (the letter $E$ refers to expressions o $L_{\mathbf{PL}}$) in the case of variables and constants. It displays the fact that semantic concepts are defined for interpreted linguistic items. We do not need to introduce **s** in the case of predicate letters, because they operate on already interpreted arguments and preserve their interpretation given by **I**. Taking **s** into account we have a simple device to distinguish models and interpretations. The latter are, so to speak, candidates for the former. It is also important that predicates are not variables (they are quantified, at least in first-order languages), but parameters. The meaning of predicates can be arbitrary, but it is always established. From a purely logical point of view, it is not important whether we preserve meanings from ordinary language and science or use our semantic imagination.•

**(D22)** (a) **s** satisfies the formula $_iP_j(t_1,..., t_i)$ in **M** iff
$<\mathbf{V}(t_1, \mathbf{s}),...,\mathbf{V}(t_i, \mathbf{s})> \in \mathbf{I}(_iP_j)$;
(b) **s** satisfies the formula $\neg A$ in **M** iff **s** does not satisfy $A$ in **M** ;
(c) **s** satisfies the formula $A \wedge B$ in **M** iff **s** satisfies $A$ in **M** and **s** satisfies $B$ **M**;
(d) **s** satisfies the formula $A \vee B$ in **M** iff **s** satisfies $A$ in **M** or **s** satisfies $B$ in **M**;
(e) **s** satisfies the formula $A \rightarrow B$ in **M** iff **s** does not satisfy $A$ in **M** or **s** satisfies $B$ **M**;
(f) **s** satisfies the formula $A \leftrightarrow B$ in **M** iff **s** satisfies $A$ in **M** and **s** satisfies $B$ in **M** or **s** does not satisfy $A$ in **M** and **s** does not satisfy $B$ in **M**;
(g) **s** satisfies the formula $\forall x_iA$ in **M** iff $A$ is satisfied in **M** by every sequence **s'** which differs from **s** at most place $i$;
(h) **s** satisfies the formula $\exists x_iA$ in **M** iff $A$ is satisfied in **M** by some sequence **s'** which differs from **s** at most at place $i$.

**Remark 18.** Infinite sequences are a technical device. Formulas of **PL** are finite strings of symbols, but they can have arbitrary lengths. If a sequence is infinite it covers strings of arbitrary finite length and allows us to formulate definition of satisfaction in a completely general manner. Finite sequences of sufficient finite length are another instrument which can be used in the present context. Without such devices, the definitions would have to be given separately for every arity On the other hand, satisfaction (or non-satisfaction) always depends only on a finite number of members of a sequence in question, more precisely on those members which correspond with free variables of a given formula.•

(**D22**) A sentence is true in **M** iff it is satisfied in **M** by every sequence.

(**D23**) *A* is a tautology of **PL** iff *A* is true in every model **M.**

**(D24) M** is a model of the set **X** of sentences iff for every *A*, **M** is a model of *A*.

(**D24**) **X** is satisfiable iff there is a model of **X**.

(**D25**) **X** is a contradiction iff is no satisiable by any model **M.**

(**D26**) **X** $\models$ *A* iff for every model **M**, if **M** is a model of **X**, then **M** is a model of *A*.

**Remark 19.** Most definitions (**D22**) – (**D26**) are parallel to those contained in (**D7**). However, it is clear that the concept of model is much more interesting in the case of **PL** than in the case of **PC**. Models of **PC** are nothing more than sequences of logical values, but models associated with **PL** are objects with a definite internal structure. **(D22)** provides famous semantic definition of truth, due to Tarski. There is a lot to say about this idea, but most of possible comments are irrelevant to metalogical themes. Let me restrict to an observation that if *A* is sentence, it is either satisfied by all sequences or by no sequence. It seems natural to define truth by satisfaction by every sequence and falsity by satisfaction by no sequence. In the case of sentences, satisfaction by every sequence is equivalent to satisfaction by one sequence and satisfaction by the empty sequence, that is, the function from ∅ to ∅. The main idea behind (**D22**) is that satisfaction of sentences is independent of any interpretation of individual variables, but only on the domain and its structure, that is, distinguished objects and the interpretation of predicates.•

## §12. Metalogical properties of PL.

All definitions (**D8**) – (**D18**) apply to **PL** and theories formalized on its basis. **PL** is recursively axiomatized, consistent (the proof goes indirectly and shows that if inconsistency of **PL** would entail inconsistency of **PC**, but the latter is consistent) and (**AX4**) is its an independent axiomatics. **PL** is, similarly as **PC**, not syntactically complete, but unlikely propositional logic, not Post-complete and above all, undecidable. If we add a sentence “there are *n* objects” which is unprovable in pure logic to axioms of **PL,** the resulting system does not lose consistency. The proof of undecidability of **PL** was one of the most important events in the history of logic. It was due to Alonzo Church (1936). The proof employs the following facts: (a) arithmetic of natural numbers is a finite extension of **PL**; (b) arithmetic of natural numbers is undecidable; (c) if **Y** is a finite extension of **X**, then **X** is undecidable, provided that **Y** is undecidable. Monadic predicate calculus is decidable (however, it is perhaps interesting to note that intuitionistic predicate calculus is undecidable). Also there are many decidable cases, for instance, formulas beginning with all universal quantifiers or with the block $\forall\exists\forall$ of quantifiers.

Our main concern is to prove soudness and semantic completeness of **PL**.

(**SPL**) (Soudness) If $\vdash A$, then $\models A$ (if $A$ is a theorem, it is a tautology).

Proof. It remains to check (**APL1**) – (**APL2**) and (**RG**) (soudness of axioms taken from **PC** was already proved). We must prove that our axioms are tautologies, but (**RG**) leads from tautologies to tautologies.

(a) $\forall x_i A \rightarrow A(t_i/x_i)$ (if the term $t_i$ is substitutable in $A$ for $x_i$).

Denote this formula by $T$. Assume that it is not tautology. So there is a model **M** and a sequence **s** such that **s** does not satisfy $T$ in **M.**. More specifically **s** satisies $\forall x_i A$, but **s** does not satisfy $A(t_i/x_i)$. (**D22**g) says that **s** satisfies the formula $\forall x_i A$ in **M** iff $A$ is satisfied in **M** by every sequence **s'** which differs from **s** at most place $i$. The sequence **s** has an object $u_i$ at the place $i$. We form a sequence take **s'** such that $u_i$ is replaced by **V**($t_i$, **s**). This new sequence differs from **s** at most at the place $i$. It satisfies the formula $\forall x_i A$ and the formula $A(t_i/x_i)$. Thus, **s** also satisfies both formulas. However, it contradicts our assumption that s does not satisfy $T$.

(b) Analoguosly for (**APL2**) and (**RG**).

**(GMPL)** (The Gödel-Malcev theorem for **PL**) Every consistent first-order theory has a model.

Proof. (A sketch) Assume that **X** is a consistent first-order theory, that is, $Cn$**X** $\subseteq$ **X** (*Cn* is defined by axioms (**C1**) – (**C12**). We add to $\mathbf{L_{LP}}$ a countably infinite set of individual constants **C** = $\{a_1, a_2, ...\}$ which do not occur in the language. This operation leads to a new theory **X'** which is a linguistic variant of **X** and also to *Cn'* such that $Cn'$**X** = **X'**. We form an infinite sequence of all formulas with at most one free variable. We can assume that indexes of formulas and their variables are the same. Thus we have the sequence of formulas **FO** = $<A_1(x_1), A_2(x_2), A_3(x_3),... >$. Now we take **C** and select from it a subsequence **C'** = $< c_1, c_2, c_3,... >$ such that no its member occurs in **FO** and such in which all members are different. Let $B_k$ be formula of the type $A_k(t_k/ x_k) \rightarrow \forall x_k A_k(x_k)$. Now consider the theory **T** = **X'** $\cup \{B_1, B_2, B_3,...\}$. It is consistent. Hence, by (**L**), it has a maximally consistent extension $\mathbf{X}^+$. The next step consists in constructing an interpretation **IN** based on terms occurring in **T** (this stage is somehow similar to the proof of the Gödel-Malcev theorem for **PC**). We take **IN** = <**W**, **Φ**>, where **W** is the set of all constants of the considered theory supplemented by $c_1, c_2, c_3,...$, and **Φ** satisfies the following conditions (a) $\Phi (P_k (t_1,..., t_k)) \leftrightarrow {}_kP_i(t_1,..., t_k)$; (b) $\Phi(c_k) \in \mathbf{X}^+$ ; . Further, we prove for **IN** the following statement (c) if $A$ is a sentence, **IN** is a model of $A$ if and only if $A \in \mathbf{X}^+$. Since $\vdash \forall x_i A$ iff $\vdash A$ and $\mathbf{X} \subseteq \mathbf{X'} \subseteq \mathbf{X}^+$, **IN** is a countable model of **X**. It completes the proof.

Further netalogical results about **PL** and first-order theories:

(**GMPL') X** is a consistent theory iff it has a model.

(**GCT**) (The Gödel completeness theorem) $\vdash A$ iff $\models A$.

(**SCTPL**) $\mathbf{X} \vdash A$ iff $\mathbf{X} \models A$.

(**COPL**) If every finite subset of a given set **X** of first-order formulas has a model, then **X** has a model.

(**LS**) (The Löwenheim-Skolem theorem) If a set **X** of first-order sentences has an infinite model, it has also a countable model.

(**LI**) (The Lindström theorem) First-order logic is the strongest logic which satisfies (**COPL**) and (**LS**).

(**NDC**) (The Grzegorczyk theorem) **PL** does not distinguish any extralogical constant or predicate, that is, something is provable in **PL** on an individual object, a property or relation being the denotation of a predicate, it is also provable in on any other extralogical object property or relation.

**Remark 20**. The **(GMPL)** is a fundamental theorem of **PL**. It is essentially used in proofs of all others metalogical results about first-order logic. The proof of this theorem due to Leon Henkin and sketched above is non-constructive. Perhaps it would be added that this matter is somehow controversial, because there is no common agreement concerning the scope of constructive methods. Another feature of **PL** is out of any doubt. Due to undecidability of **PL**, this system is less effective than **PC**. Thus, **PL** is not perfectly symmetric with respect to syntax and semantics.•

### §13. Remarks on the concept of logic in the light of metalogic.

Remarks in this section have a philosophical character and come back to some problems indicated in §1. What insights about logic itself are suggested by metalogic? Traditionally, logic is considered as general, topically-neutral and independent of any particular assumptions. In fact, these features are various aspects of the same property which can be termed as universality. It is obvious that metalogical properties of **PC** and **PL** display the universality of logic. As it was already noted logic can be defined by the formula (all remarks are addressed to classical logic, but they can be extended to non-classical systems as well):

(**LO**) **LOG** = $Cn\varnothing$.

This definition looks artificial at first sight, because it is clear that the logical content is related to axioms imposed on $Cn$; clearly, the empty set here is a convenient metaphor: we can derive something from the empty set only because of the logical machinery already built into $Cn$. First of all, let us observe that (**LO**) is equivalent to two other statements, namely:

(**LO'**) $A \in$ **LOG** iff $\neg A$ is inconsistent.

(**LO''**) **LOG** is the only non-empty product of all consistent deductive systems (theories).

Now, (**LO'**) and (**LO''**) surely define properties which we expected to be possessed by any logic. We agree that negations of logical principles are inconsistencies and that logic is the common part of all, even mutually, inconsistent theories. Additionally, (**LO''**) entails that logical laws are derivable from arbitrary premises. Thus, we have: $A \in Cn\varnothing$ iff $A \in Cn\mathbf{X}$, for any **X**, and the equality **LOG** =

$Cn\varnothing = Cn$**X**, for any **X**. These considerations show that (**LO**) and its equivalents express an important intuition, namely that logic is universal in the sense that it does not require any premises, or is deducible from arbitrary assumptions.

Yet one might argue that such a construction of logic is circular because it defines logic by means of the prior assumption that something is logical. This objection can be easily met by pointing out that our definitions are inductive, that is, selects logical axioms as so called initial conditions and then shows how inductive conditions (in fact, the rules of inference coded by *Cn*) lead step by step to new logical elements. On the other hand, it is perhaps important for philosophical reasons to look at an independent characterization of logic. This is provided by semantics and it is expressed by):

(**LO'''**) $A \in$ **LOG** if and only if for every model **M.**, *A* is true in **M**.

This last definition describes logic as universal in the sense that logical laws are true in every model (domain). It is related to the old intuition that logic is topic neutral, that is, true or valid with respect to any particular subject matter. Intuitively, there is an obvious link between (**LO**)-(**LO''**) and (**LO'''**) which is formally captured by the weak completeness theorem for **PL**. Another aspect of the universality of logic is displayed by (**NDC**). It is clear that the topic-neutrality of logic consists in its independence of individuals and their properties. The role of the weak completeness theorem in defining logic is clear. However, this fact cannot be exaggerated. The content of this theorem strongly depends of properties ascribed to $\vdash$ or *Cn*. Unfortunately, there is no clear idea how to delineate the stock of logical constants, that is, divide all terms into logical and extralogical. Are second-order quantifiers "logical"? Are modal notions logical? Is the membership relation "logical"? There even doubts concerning identity, because it allows us to define numerical quantifiers (there are *n* objects?) which rather express extralogical facts?

There is a considerable debate concerning the interpretation and consequences of (**LI**). All parties agree that (**LI**) asserts the limitations on the expressive power of first-order predicate logic. In particular, several mathematical concepts, like finiteness, cannot be defined in its language. However, it is a matter of controversy whether (**LI**) determines that only first-order predicate logic deserves to be counted as *the* logic. The first-order thesis just restricts the scope of logic to first-order logic, but the opposite standpoint maintains that if logic is to serve mathematics, its expressive power must be much greater than that of first-order languages. It is now clear why this problem becomes central when an extended concept of logic is assumed. Since definability is traditionally regarded as a logical issue, its limitations are perceived as limitations of the power of logic. On the other hand, it is now clear in the light of metalogic, that first-order logic is the only logic which is fully universal. Other systems introduce several extralogical features. It would be too optimistic to expect that metalogic could solve the deepest philosophical problems concerning logic. However, it is certainly true that metalogical results help in a better understanding of the philosophy of logic.

## Bibliograpical hints

Almost every modestly advanced textbook of logic considers metalogical topics. The following books are particularly devoted to metalogic and they also contain relevant historical information:

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J. Barwise & S. Feferman (Eds.). **Model-Theoretic Logics**, Berlin: Springer-Verlag 1985.

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