Assignments I

T1. Consider a dichotomous property of individuals in a large finite population. Let the population size be N and the size of a sample taken without replacement be n. This situation corresponds to an urn containing N balls, which are either black or white. Let θ denote the number of black balls in the urn. The probability of a sample of n balls contains x black ones is given by the hypergeometric expression

$$p(x|\theta) = \frac{\binom{\theta}{x}\binom{N-\theta}{n-x}}{\binom{N}{n}}.$$

Further, if $p(\theta = r), r = 0, ..., N$, specifies the prior probabilities for θ, we get the posterior probability of $\theta = N$ as

$$p(\theta|x = n) = \frac{p(x = n|\theta = N)p(\theta = N)}{\sum_{r=n}^{N} p(x = n|\theta = r)p(\theta = r)}.$$

Assume a uniform prior and calculate explicitly the above posterior probability. Analyze the situation from a general scientific perspective. Hint: you can utilize the following general result

$$\sum_{r=0}^{N} \binom{r}{l}\binom{N-r}{m} = \binom{N+1}{l+m+1},$$

where $l + m = n$ is the sample size for the hypergeometric model and l stands for the number of picked individuals having the characteristic of interest (*e.g.* black colour).

T2. You observe two thumbtack tosses which both land with the point down. Investigate the predictive distribution for the outcome of the next toss, using the priors: Beta(1/2,1/2), Beta(1,1),Beta(0,0),Beta(10,10). How does the prior affect your predictions? Hint: The predictive distribution is a Beta-Binomial distribution, which can be analytically defined through the Beta-integral.

T3. Consider a vector obtained by norming the vector $\lambda = (\phi, 1), \phi \in (-\infty, \infty)$, into half unit circle (the length of the vector is one). In the polar coordinates the vector λ is $(\cos\theta, \sin\theta), \theta \in [0, \pi)$. If θ is given the uniform prior, which prior is implied for ϕ? Hint: $\theta = \operatorname{arccot}\phi$. If you wish, you can also consider the distribution of θ, implied by a "uniform" distribution for ϕ.

T4. In betting situations one is often interested in odds, referring in the thumbtack tossing situation to the quantity $\theta/(1-\theta)$. Alternatively one may consider the log-odds

$$\lambda = \log\frac{\theta}{1-\theta}.$$

Show that a ”uniform” distribution for λ implies the following distribution for θ

$$p(\theta) = \theta^{-1}(1-\theta)^{-1}.$$

What problems are associated with this distribution if it is used as a prior in the thumbtack tossing problem?